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Jochen Wengenroth

# Derived Functors in Functional Analysis

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Author

Jochen Wengenroth
FB IV - Mathematik
Universität Trier
54286 Trier, Germany

*E-mail: wengen@uni-trier.de*

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# 1

# Introduction

In the last years, the part of functional analysis which contributes to the solution of analytical problems using various techniques from the theory of locally convex spaces gained a lot of strength from new developments in topics which are related to category theory and homological algebra. In particular, progress about the derived projective limit functor (which measures the obstacle against the construction of a global solution of a problem from local solutions) and the splitting theory for Fréchet and more general spaces (which is concerned with the existence of solution operators) allowed new applications for instance to problems about partial differential or convolution operators.

The connection between homological algebra and the theory of locally convex spaces had been established by V.P. Palamodov [50] in 1969. He pointed out that a number of classical themes from functional analysis can be viewed as exactness problems in appropriate categories and thus can be investigated with the aid of derived functors. After developing suitable variants of tools from category theory he constructed the derivatives of a fairly wide class of functors and proved concrete representations, characterizations and relations for several functors acting on the category of locally convex spaces, like the completion, duality or Hom-functors. A major role in these investigations was played by the projective limit functor assigning to a countable projective limit of locally convex spaces its projective limit. A very detailed study of this functor was given by Palamodov in [49].

Starting in the eighties, D. Vogt reinvented and further developed large parts of these results in [62] (which never had been published) and [61, 63, 64, 65] with a strong emphasis on the functional analytic aspects and avoiding most of the homological tools. He thus paved the way to many new applications of functional analytic techniques. Since then, the results (in particular about the projective limit functor) have been improved to such an extent that they now constitute a powerful tool for solving analytical problems.

The aims of this treatise are to present these tools in a closed form, and on the other hand to contribute to the solution of problems which were left open in Palamodov's work [50, §12]. We try to balance between the homological

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viewpoint, which often illuminates functional analytic results, and techniques from the theory of locally convex spaces, which are easier accessible for the typical reader we have in mind. Therefore we assume a good familiarity with functional analysis as presented e.g. in the books of Bonet and Pérez-Carreras [51], Jarchow [36], Köthe [39], or Meise and Vogt [45]. Except for some examples we will not need anything beyond these text books. On the other hand, no knowledge about homological algebra is presumed. Chapter 2 reviews the definitions and results (including some ideas for the proofs) that will be used in the sequel. This is only a small portion of the material presented and needed in Palamodov's work. Readers who are interested in the relation of topological vector spaces to more sophisticated concepts of category theorey may consult the articles [52, 53] of F. Prosmans.

The key notions in chapter 2 are that of short exact sequences in suitable categories (for instance, in the category of locally convex spaces

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is an exact sequence if $f$ is a topological embedding onto the kernel of $g$ which is a quotient map) and the notion of an additive functor which transforms an object $X$ into an object $F(X)$ and a morphism $f : X \to Y$ into a morphism $F(X) \longrightarrow F(Y)$. The derived functors are used to measure the lack of exactness of the complex

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0.$$

If the values $F(X)$ are abelian groups or even vector spaces then exactness of the sequence means that $F(f)$ is injective, its image is the kernel of $F(g)$, and $F(g)$ is surjective. For example, if $E$ is a fixed locally convex space and $F$ assigns to every locally convex space $X$ the vector space $\mathrm{Hom}(E, X)$ of continuous linear maps and to $f : X \to Y$ the map $T \mapsto f \circ T$, then the exactness of the sequence above means that each operator $T : E \to Z = Y/X$ has a lifting $\tilde{T} : E \to Y$.

If the functor $F$ has reasonable properties, one can construct derived functors $F^k$ such that every exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is transformed into an exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow F^1(X) \longrightarrow F^1(Y) \longrightarrow \dots$$

Then $F^1(X) = 0$ means that

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow 0$$

is always exact.

Chapter 3 develops the theory of the countable projective limit functor starting in 3.1 with a "naive" definition of the category of projective spectra where the objects $\mathscr{X} = (X_n, \varrho^n_{n+1})$ consist of linear spaces $X_n$ and linear spectral maps $\varrho^n_{n+1}$, and the morphisms $f = (f_n : X_n \longrightarrow Y_n)_{n\in\mathbb{N}}$ consist of linear maps commuting with the spectral maps. This definition differs from the one given by Palamodov but has the advantage of being very simple. The functor Proj (which is also denoted by $\varprojlim$ in the literature) then assigns to a spectrum $\mathscr{X}$ its projective limit

$$X = \mathrm{Proj}\mathscr{X} = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \varrho^n_{n+1}(x_{n+1}) = x_n \right\}$$

and to a morphism $f$ the linear map $\mathrm{Proj}(f) : (x_n)_{n\in\mathbb{N}} \mapsto (f_n(x_n))_{n\in\mathbb{N}}$. If we consider the "steps" $X_n$ as the local parts of $X$ and we are concerned with the problem whether a given map $f^* : X \to Y$ is surjective, we can try to solve the problem locally which requires to find a morphism $f$ with surjective components $f_n : X_n \to Y_n$ such that $f^* = \mathrm{Proj}(f)$, and then we can hope to conclude the surjectivity of $f^*$ which requires knowledge about $\mathrm{Proj}^1\mathscr{K}$ where $\mathscr{K}$ is the spectrum consisting of the kernels $\ker f_n$.

After presenting the homological features of this functor and comparing its applicability with Palamodov's original definition, we give in section 3.2 a variety of characterizations and sufficient conditions for $\mathrm{Proj}^1\mathscr{X} = 0$. The unifying theme of all these results is the Mittag-Leffler procedure: one seeks for corrections in the kernels of the local solutions which force the corrected solutions to converge to a global solution. If the steps of the spectrum are Fréchet spaces this idea leads to a characterization of $\mathrm{Proj}^1\mathscr{X} = 0$ due to Palamodov. We present three proofs of this which stress different aspects and suggest variations in several directions. One of the proofs reduces the result to the classical Schauder lemma which is a version of the open mapping theorem. It is this proof which easily generalizes to a theorem of Palamodov and Retakh [50, 54] about $\mathrm{Proj}^1\mathscr{X} = 0$ for spectra consisting of $(LB)$-spaces and clarifies the role of the two conditions appearing in that theorem: the first is the continuity and the second is the density required for the Mittag-Leffler procedure. Knowing this, it is very surprising that in many cases the theorem remains true without the first assumption. The argument behind is again a version of the Schauder lemma (even a very simple one). This trick tastes a bit like lifting oneself by the own bootstraps, but in our case it works. After discussing this circle of results with an emphasis on spectra consisting of $(LS)$-spaces, we consider in section 3.3 topological consequences (like barrelledness conditions and quasinormability) for a projective limit if some representing spectrum satisfies $\mathrm{Proj}^1\mathscr{X} = 0$, and we solve one of Palamodov's questions about Proj considered as a functor with locally convex spaces as values: the algebraic property $\mathrm{Proj}^1\mathscr{X} = 0$ does not imply topological exactness in general, but it does indeed under an additional assumption which is satisfied in all situations which appear in analysis.

Section 3.4 contains some applications of the results obtained in 3.2 and 3.3. We start with some very classical situations like the Mittag-Leffler theorem or the surjectivity of $\overline{\partial}$ on $\mathscr{C}^\infty(\Omega)$ for open set $\Omega \subseteq \mathbb{C}$. The techniques based on the projective limit functor nicely separate the two aspects of the standard proofs into a local and a global part. We also give a proof of Hörmander's characterization of surjective partial differential operators on $\mathscr{D}'(\Omega)$ and finally explain results of Braun, Meise, Langenbruch, and Vogt about partial differential operators on spaces of ultradifferentiable functions.

Encouraged by the results of chapter 3 and the simple observation that every complete locally convex space is the limit of a projective spectrum of Banach spaces (which is countable only for Fréchet spaces), we investigate in chapter 4 the homological behaviour of arbitrary projective limits. In a different context, this functor has been investigated e.g. by C.U. Jensen [37]. In section 4.1 the algebraic properties are developed similarly as in 3.1 for the countable case, and we present Mitchell's [47] generalization of the almost trivial fact that $\mathrm{Proj}^k\mathscr{X} = 0$ for $k \geq 2$ and countable spectra: if $\mathscr{X}$ consists of at most $\aleph_n$ objects (in our case linear spaces) then $\mathrm{Proj}^k\mathscr{X} = 0$ for $k \geq n+2$.

Before we consider spectra of locally convex spaces, we insert a short section about the completion functor with a result of Palamodov and a variant due to D. Wigner [72] who observed a relation between the completion functor and the derivatives of the projective limit functor which is presented in 4.3. Besides this, we prove a generalization of Palamodov's theorem about reduced spectra $\mathscr{X}$ of Fréchet spaces in the spirit of Mitchell's result mentioned above: if $\mathscr{X}$ consists of at most $\aleph_n$ spaces then $\mathrm{Proj}^k\mathscr{X} = 0$ holds for $k \geq n+1$. This seems to be the best possible result: using ideas of Schmerbeck [55], we show that under the continuum hypothesis (in view of the result above this set-theoretic assumptions appears naturally) the canonical representing spectrum of the space $\varphi$ of finite sequences endowed with the strongest locally convex topology satisfies $\mathrm{Proj}^k\mathscr{X} = 0$ for $k \geq 2$ but $\mathrm{Proj}^1\mathscr{X} \neq 0$. The same holds for all complete separable (DF)-spaces satisfying the "dual density condition" of Bierstedt and Bonet [6] (this is the only place where we use arguments of [51] which do not belong to the standard material presented in books about locally convex spaces). These negative results lead to a negative answer to another of Palamodov's questions. The essence of chapter 4 is that the first derived projective limit functor for uncountable spectra hardly vanishes (we know essentially only one non-trivial example given in 4.1) and that this theory is much less suitable for functional analytic applications than in the countable case.

In chapter 5 the derivatives $\mathrm{Ext}^k(E,\cdot)$ of the functors $\mathrm{Hom}(E,\cdot)$ are introduced, and we explain the connection to lifting, extension, and splitting properties (it is this last property which is used to find solution operators in applications). We show that for a Fréchet space $X$ there is a close relation between $\mathrm{Ext}^k(E,X)$ and $\mathrm{Proj}^k\mathscr{Y}$ for a suitable spectrum $\mathscr{Y}$ and use this to give a simplified proof of the fact that $\mathrm{Ext}^k(E,X) = 0$ for all $k \geq 1$ whenever $E$ is a complete (DF)-space and $X$ is a Fréchet space and one of them is nuclear (this may serve as a guide for the case of two Fréchet spaces considered in 5.2). The rest of section 5.1 is devoted to a conjecture of Palamodov that under the same assumptions for $E$ and $X$ also $\mathrm{Ext}^k(X, E) = 0$ holds. The only Fréchet space $X$ for which we can provide some information is $X = \omega = \mathbb{K}^{\mathbb{N}}$. For this case, we could show jointly with L. Frerick that $\mathrm{Ext}^1(\omega, E) = 0$ for "most" (DF)-spaces. On the other hand, the negative results of chapter 4 eventually lead to $\mathrm{Ext}^2(\omega, \varphi) \neq 0$ at least under the continuum hypothesis.

In 5.2 we present Vogt's [63] arguments which led to a fairly complete characterization of $\mathrm{Ext}^1(E, F)$ for pairs of Fréchet spaces in [29]. We deduce from the splitting theorem the most important results about the structure of nuclear Fréchet spaces (which are due to Vogt [59] and Vogt and Wagner [67]) to compare these with results in 5.3 about splitting in the category of (PLS)-spaces (in particular, spaces of distributions). We first present very recent results of P. Domański and Vogt [24, 25] about the structure of complemented subspaces of $\mathscr{D}'$ (with only minor modifications of their proof, but having the aesthetical advantage of staying in the category of (PLS)-spaces) and deduce from this an improvement of their result about $\mathrm{Ext}^1_{PLS}(E, F) = 0$ which shows that $\mathscr{D}'$ plays exactly the same role for splitting in the category of (PLS)-spaces as $s$ does for nuclear Fréchet spaces. This has immediate applications for the splitting of distributional complexes.

In the sixth chapter about inductive limits we explain the relation to the projective limit functor which gives several characterizations of acyclic (LF)-spaces. We provide a very short proof for the completeness of these spaces and show that for (LF)-spaces acyclicity is equivalent to many regularity conditions of the inductive limit. Because of the close connection to projective spectra of (LB)-spaces and in view of existing literature about inductive limits (in particular the book of Bonet and Pérez-Carreras [51]) this discussion is rather short. The rest of the chapter is devoted to questions of Palamodov whether inductive limits of complete locally convex spaces are always complete and regular. We provide positive answers under a very weak extra assumption.

The final chapter is devoted to the duality functor assigning to a locally convex space its strong dual and to a continuous linear map the transposed operator. For an exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

of locally convex spaces neither $f^t$ nor $g^t$ need be open onto its range.

This "lack of openness" is measured by the derived functors $D^+(X)$ and $D^1(X)$, respectively. We derive this characterization from the homological definitions and provide a quite simple proof of a result due to Palamodov [50], Merzon [46], and Bonet and S. Dierolf [8] characterizing the quasinormable Fréchet spaces by $D^1(X) = 0$ and a lifting property for bounded sets, where again the Schauder lemma plays the main role. Moreover, we show that beyond the class of Fréchet spaces quasinormability is not sufficient for vanishing of $D^+$ nor of $D^1$ (we suspect that these answers to further questions of Palamodov were probably known to many people for quite a while). We finish with a surprisingly general positive result about the (topological) exactness of

$$0 \longrightarrow Z'_\beta \xrightarrow{g^t} Y'_\beta \xrightarrow{f^t} X'_\beta \longrightarrow 0,$$

where the strict Mackey condition (which is dual to quasinormability) enters the game, and apply this to projective limits of (LB)-spaces.

As we said above, a good portion of this treatise (in particular chapter 4 and partly 5.1, 6, and 7) is motivated by the list of unsolved problems in Palamodov's work. These parts are probably much less important for applications than other parts. But one should keep in mind that the efforts for searching counterexamples led to several positive results which allow applications.

In this work we touch various fields of the theory of locally convex spaces which have quite a long tradition. It would have been expedient or even necessary to explain the background of many results with much more care. I refrained from really trying to do so because this would have changed the character of this work and because there are many people who are much better qualified for this.

# 2

# Notions from homological algebra

In this chapter we recall very briefly some definitions and basic results from category theory and homological algebra. For a reader who is unfamiliar with these notions it might be helpful to translate the abstract definitions into concrete ones e.g. for the category of vector spaces.

## 2.1 Derived Functors

An additive category $\mathscr{K}$ consists of a class of objects and abelian groups $\mathrm{Hom}(X,Y)$ of morphisms for each pair of objects (the group operation is always denoted by $+$ and the neutral element by $0$) together with a composition $\circ : \mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(X,Z)$ which is associative and distributive, i.e. $(f+g)\circ h = f\circ h + g\circ h$ and $f\circ(g+h) = f\circ g + f\circ h$. Moreover it is assumed that for each object $X$ there is $id_X \in \mathrm{Hom}(X,X)$ with $id_X \circ f = f$ and $f \circ id_X = f$, and that there is an object $0$ with $|\mathrm{Hom}(X,0)| = |\mathrm{Hom}(0,X)| = 1$ for all objects $X$.

The symbols $f : X \longrightarrow Y$, $X \stackrel{f}{\longrightarrow} Y$, $Y \stackrel{f}{\longleftarrow} X$ all mean that $X$ and $Y$ are objects of the category under considerations and $f \in \mathrm{Hom}(X,Y)$.

$f : X \longrightarrow Y$ is a monomorphism (epimorphism, respectively) if $f \circ g = 0$ ($h \circ f = 0$) implies $g = 0$ ($h = 0$). $f$ is called bimorphism if both conditions hold, and an isomorphism if there is $g : Y \longrightarrow X$ with $f \circ g = id_Y$ and $g \circ f = id_X$.

A kernel of $f : X \longrightarrow Y$ is a monomorphism $\ker f \stackrel{k}{\longrightarrow} X$ with $f \circ k = 0$ and the universal property that $f \circ j = 0$ for some $j : Z \longrightarrow X$ implies the existence of a (unique) $\tilde{\jmath} : Z \longrightarrow \ker f$ with $j = k \circ \tilde{\jmath}$. Dually, a cokernel of $f$ is an epimorphism $Y \stackrel{c}{\longrightarrow} \operatorname{coker} f$ which is universal with respect to the condition $c \circ f = 0$, i.e. $q \circ f = 0$ for some $q : Y \longrightarrow Z$ implies $q = \tilde{q} \circ c$ for a unique $\tilde{q} : \operatorname{coker} f \longrightarrow Z$. Obviously, two kernels $k : \ker f \to X$ and $\tilde{k} : K \to X$ are isomorphic, i.e. there is an isomorphism $j : \ker f \to K$ with $\tilde{k} \circ j = k$ and the same applies to cokernels. We will therefore often speak

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about *the* kernel (cokernel) of a morphism if we mean some and it is clear that each has the property under consideration (which always follows from the isomorphy). Moreover, we will sometimes call the object ker$f$ (coker $f$) the kernel (cokernel) of $f$.

An image $i : \mathrm{im}\, f \longrightarrow Y$ of $f : X \longrightarrow Y$ is a kernel of the cokernel of $f$ and a coimage $q : X \longrightarrow \mathrm{coim}\, f$ is a cokernel of the kernel of $f$.

If $f$ has an image and a coimage then there is a unique morphism $\tilde{f}$ : $\mathrm{coim}\, f \longrightarrow \mathrm{im}\, f$ such that

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle q}\big\downarrow & & \big\uparrow{\scriptstyle i} \\ \mathit{coim} f & \xrightarrow{\ \tilde{f}\ } & \mathit{im} f \end{array}$$

commutes, i.e. $i \circ \tilde{f} \circ q = f$. $f$ is called a homomorphism if $\tilde{f}$ is an isomorphism (which does not depend on the choice of the image and coimage). It is easy to check that a kernel or a cokernel of a morphism is automatically a homomorphism.

An additive category $\mathscr{K}$ is called semi-abelian if every morphism $f$ has a kernel and a cokernel and the induced morphism $\tilde{f}$ as above is a bimorphism. Moreover, it is assumed that for each pair $(X_1, X_2)$ of objects there exists a product, i.e. an object $P$ and morphisms $\pi_j : P \longrightarrow X_j$ such that for all $g_j : Y \longrightarrow X_j$ there is a unique $g : Y \longrightarrow P$ with $g_j = \pi_j \circ g$. If in addition, every morphism is a homomorphism $\mathscr{K}$ is called abelian.

An object $I$ of a semi-abelian category $\mathscr{K}$ is injective if for every $f$ : $X \longrightarrow I$ and every monohomomorphism $i : X \longrightarrow Y$ there is an "extension" $\tilde{f} : Y \longrightarrow I$ with $\tilde{f} \circ i = f$. An object $P$ is projective if for every $f : P \longrightarrow X$ and every epihomomorphism $q : Y \longrightarrow X$ there is a "lifting" $\tilde{f} : P \longrightarrow Y$ with $q \circ \tilde{f} = f$. The category is said to have many injective objects if for every object $X$ there are an injective object $I$ and a monohomomorphism $i : X \longrightarrow I$. A sequence $\ldots \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \ldots$ is called complex if $g \circ f = 0$. The complex is called acyclic at $Y$ if the image of $f$ is the kernel of $g$ (as we have said above, this means that every image of $f$ is a kernel of $g$) and left exact (or right exact or exact) at $Y$ if in addition $f$ (or $g$ or both) is a homomorphism.

A co- (or contra-) variant functor $F$ from a category $\mathscr{K}$ to another category $\mathscr{A}$ is a rule (depending on the underlying set theory one might also say a map) assigning to each object $X$ of $\mathscr{K}$ an object $F(X)$ of $\mathscr{A}$ and to each morphism $f : X \to Y$ a morphism $f^* = F(f) : F(X) \longrightarrow F(Y)$ ($f_* : F(Y) \to F(X)$, respectively) such that $F(f \circ g) = F(f) \circ F(g)$ ($F(f \circ g) = F(g) \circ F(f)$). The following definitions are given for covariant functors only, the modifications needed for contravariant functors are obvious. $F$ is called additive if $\mathscr{K}$ and $\mathscr{A}$ are additive categories and $F(f+g) = F(f)+F(g)$ holds for all $f, g : X \longrightarrow Y$.

In that case, complexes

$$\ldots \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \ldots$$

are transformed into complexes

$$\ldots \longrightarrow F(X) \xrightarrow{f^*} F(Y) \xrightarrow{g^*} F(Z) \longrightarrow \ldots$$

Derived functors are used to study which exact complexes remain exact when the functor is applied.

Let $\mathscr{K}$ be a semiabelian category and $F$ a covariant additive functor from $\mathscr{K}$ to an abelian category $\mathscr{A}$. $F$ is called injective if it transforms exact complexes

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$$

(this means that the complex is exact at $X$ and $Y$) into exact complexes

$$0 \longrightarrow F(X) \xrightarrow{f^\star} F(Y) \xrightarrow{g^\star} F(Z).$$

Equivalently, $F$ is injective if it transforms the kernel of every homomorphism $g : Y \longrightarrow Z$ into the kernel of $g^*$. $F$ is called semi-injective if exact complexes

$$0 \longrightarrow X \xrightarrow{f} Y$$

are transformed into exact complexes

$$0 \longrightarrow F(X) \xrightarrow{f^\star} F(Y),$$

i.e. monohomomorphisms are transformed into monohomomorphisms.

Let now $\mathscr{K}$ be a semi-abelian category with many injective objects. Then every object $X$ has an injective resolution

$$(I) \qquad 0 \longrightarrow X \xrightarrow{i} I_0 \xrightarrow{i_0} I_1 \xrightarrow{i_1} I_2 \longrightarrow \ldots,$$

i.e. an exact complex with injective objects $I_0, I_1, \ldots$ Such an injective resolution can be constructed inductively, starting with a monohomomorphism $i : X \to I_0$, then choosing a monohomomorphism $j_0 : \text{coker } i \longrightarrow I_1$, and so on. If $F$ is a covariant semi-injective functor from $\mathscr{K}$ to an abelian category $\mathscr{A}$, the derived functors are defined as the cohomology of the complex

$$0 \longrightarrow F(I_0) \xrightarrow{i_0^*} F(I_1) \xrightarrow{i_1^*} F(I_2) \xrightarrow{i_2^*} \ldots,$$

this is $F^0(X) = \ker\, (i_0^*)$ and $F^k(X) = \text{coker } (\text{im } i_{k-1}^* \longrightarrow \ker\, i_k^*)$ (if $\mathscr{A}$ is the category of abelian groups one gets $F^k(X) = \ker\, i_k^*/\text{im } i_{k-1}^*$ which looks more familiar).

Moreover, $F^+(X) = \text{coker } (F(X) \longrightarrow \ker\, i_0^*)$ is called the additional derived functor (which vanishes if $F$ is an injective functor).

The derivatives do not depend on the injective resolution chosen for $X$. Indeed, if

$$0 \longrightarrow X \xrightarrow{j} J_0 \xrightarrow{j_0} J_1 \xrightarrow{j_1} \dots$$

is another injective resolution then there are $f_n : I_n \longrightarrow J_n$ and $g_n : J_n \longrightarrow I_n$ (which are canonically constructed using the injectivity of $J_n$ and $I_n$) such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{i} & I_0 & \longrightarrow & I_1 & \longrightarrow & \cdots \\
 & & \Vert & & f_0 \downarrow \uparrow g_0 & & f_1 \downarrow \uparrow g_1 & & \\
0 & \longrightarrow & X & \xrightarrow{j} & J_1 & \longrightarrow & J_1 & \longrightarrow & \cdots
\end{array}$$

is a commutative diagram from which one obtains that the cohomologies coincide. If

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is an exact complex (i.e. exact at all positions $X$, $Y$, and $Z$, we will also say exact or short exact sequence to such a complex) we can choose injective resolutions $(I)$, $(J)$, and $(K)$ for $X$, $Y$, and $Z$ such that there is a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow i & & \downarrow j & & \downarrow k & & \\
0 & \longrightarrow & I_0 & \xrightarrow{f_0} & J_0 & \xrightarrow{g_0} & K_0 & \longrightarrow & 0 \\
 & & \downarrow i_0 & & \downarrow j_0 & & \downarrow k_0 & & \\
0 & \longrightarrow & I_1 & \xrightarrow{f_1} & J_1 & \xrightarrow{g_1} & K_1 & \longrightarrow & 0 \\
 & & \downarrow i_1 & & \downarrow j_1 & & \downarrow k_1 & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

(starting with $I_0$ and $K_0$ one can take $J_0 = I_0 \times K_0$ to obtain the second row, and then the following rows are constructed inductively).

Applying the functor $F$ then gives the commutative diagram in $\mathscr{A}$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(X) & \xrightarrow{f^*} & F(Y) & \xrightarrow{g^*} & F(Z) & & \\
 & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle k^*} & & \\
0 & \longrightarrow & F(I_0) & \xrightarrow{f_0^*} & F(J_0) & \xrightarrow{g_0^*} & F(K_0) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_0^*} & & \downarrow{\scriptstyle j_0^*} & & \downarrow{\scriptstyle k_0^*} & & \\
0 & \longrightarrow & F(I_1) & \xrightarrow{f_1^*} & F(J_1) & \xrightarrow{g_1^*} & F(K_1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle j_1^*} & & \downarrow{\scriptstyle k_1^*} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

where all rows but the first one are exact. In all cases where we will apply this construction $\mathscr{A}$ is the category of vector spaces. Then one can construct by "diagram chasing" connecting morphisms $\delta^k : F^k(Z) \longrightarrow F^{k+1}(X)$ in a canonical way. For instance, for $z \in F^0(Z) = \ker k_0^*$ there is $y \in F(J_0)$ with $g_0^*(y) = z$. Since $j_0^*(y) \in \ker g_1^* = \operatorname{im} f_1^*$ there is $x \in F(I_1)$ with $j_0^*(y) = f_1^*(x)$ and $i_1^*(x) = 0$ because

$$f_2^* \circ i_1^*(x) = j_1^* \circ f_1^*(x) = j_1^* \circ j_0^*(x) = 0 \text{ and } f_2^* \text{ is injective.}$$

We define $\delta^0(z) = x + \operatorname{im} i_0^* \in F^1(X)$. It is a matter of straightforward calculations to check that this definition is independent of the choice of $x$, that $\delta^0 : F^0(Z) \longrightarrow F^1(X)$ is linear, and that

$$F^0(Y) \longrightarrow F^0(Z) \xrightarrow{\delta^0} F^1(X) \longrightarrow F^1(Y)$$

is exact at $F^0(Z)$ and $F^1(X)$. These are, in a nutshell, the arguments leading to the fundamental theorem of this chapter.

**Theorem 2.1.1** *Let $\mathscr{K}$ be a semi-abelian category with man injective objects and $F$ a covariant semi-injective functor from $\mathscr{K}$ to an abelian category $\mathscr{A}$. If*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

*is an exact complex in* $\mathscr{K}$ *then there is an exact complex in* $\mathscr{A}$

$$0 \longrightarrow F^0(X) \longrightarrow F^0(Y) \longrightarrow F^0(Z) \xrightarrow{\delta^0} F^1(X) \longrightarrow$$

$$\longrightarrow F^1(Y) \longrightarrow F^1(Z) \xrightarrow{\delta^1} F^2(X) \longrightarrow \ldots$$

By the definition of $F^+$, there are also exact sequences

$$0 \longrightarrow F(X) \longrightarrow F^0(X) \longrightarrow F^+(X) \longrightarrow 0.$$

In addition to the long cohomology sequence of the theorem above, Palamodov obtained a kind of connecting exact sequence involving $F^+(X)$. We will not use this less canonical sequence. Instead, we will give simple direct proofs in the particular situations occurring in chapters 3 and 7.

Let us note however that $F^+(Z) = 0$ in the situation of 2.1.1 holds if $Y$ is an injective object. This follows again by diagram chasing from the diagram preceding 2.1.1 since for injective $Y$ the middle column of that diagram is exact.

Let us finish this notational section by translating the homological definitions for the category of vector spaces (over a fixed field) as objects and linear maps as morphisms. Of course, the composition of morphisms is the usual composition of maps and the group operation on $\mathrm{Hom}(X, Y)$ is the addition $(f + g)(x) = f(x) + g(x)$. A linear map $f : X \longrightarrow Y$ is a monomorphism (epimorphism) iff it is injective (surjective), its kernel is the embedding of $f^{-1}(\{0\})$ into $X$, and the cokernel is the quotient map $Y \longrightarrow Y/f(X)$. Every morphism is a homomorphism and a complex

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact at $Y$ iff $f(X) = g^{-1}(\{0\})$ holds. Finally, every object is injective as well as projective which is easily seen using Hamel bases.

## 2.2 The category of locally convex spaces

The category $\mathcal{LCS}$ consists of (not necessarily Hausdorff) locally convex spaces (l.c.s.) over the same scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ as objects and continuous linear maps (operators) as morphisms. Sometimes $\mathrm{Hom}(X,Y)$ is also denoted by $L(X,Y)$ and the group structure is the usual addition of operators.

Throughout this work, we will use the standard terminology from the theory of locally convex spaces as e.g. in [36, 39, 51], in particular, for a locally convex spaces $X$ we denote by $\mathscr{U}_0(X)$ the filter basis of absolutely convex neighbourhoods of 0 and by $\mathscr{B}(X)$ the family of all absolutely convex bounded sets.

$f : X \longrightarrow Y$ is a monomorphism (epimorphism) iff it is injective (surjective). Note that this would be different if we considered only Hausdorff l.c.s., then every $f$ with dense range would be an epimorphism. Although the category of Hausdorff l.c.s might look more natural at the first sight it is worse than $\mathcal{LCS}$ because there are much less homomorphisms (each homomorphism in the category of separated l.c.s. has closed range).

In $\mathcal{LCS}$, the kernel of $f : X \longrightarrow Y$ is the identical map $f^{-1}(\{0\}) \longrightarrow X$, $x \mapsto x$ where $f^{-1}(\{0\})$ is endowed with the topology induced by $X$, and $q : Y \longrightarrow Y/f(X)$ is the cokernel of $f$, where $Y/f(X)$ is endowed with the quotient topology. Accordingly, $X/f^{-1}(\{0\})$ is the coimage of $f$ and the subspace $f(X)$ of $Y$ is the image (as we did here, we will use terminology from homological algebra and the theory of locally convex spaces a bit loosely as long as there is no danger of misunderstanding). $f$ is a homomorphism if and only if it is open onto its range. All this would be the same in the category of topological vector spaces. The fundamental difference is the presence of many injective objects in $\mathcal{LCS}$ which follows from the Hahn-Banach theorem.

**Theorem 2.2.1** *The category of locally convex spaces is semi-abelian and has many injective objects.*

**Proof.** Let us first show that for each set $M$ the Banach space

$$\ell_M^\infty := \{(x_i)_{i\in M} \in \mathbb{K}^M : \|(x_i)_{i\in M}\|_\infty := \sup_{i\in M} |x_i| < \infty\}$$

is an injective object of $\mathcal{LCS}$. If $f : X \longrightarrow \ell_M^\infty$ is a morphism and $j : X \longrightarrow Y$ is a monohomomorphism (i.e. a topological embedding) there is $U \in \mathscr{U}_0(X)$ such that $f_i \in U^\circ$ for all $i \in M$, where $f_i : X \longrightarrow \mathbb{K}$ is the composition of $f$ with the projection onto the $i$-th component. If $V \in \mathscr{U}_0(Y)$ satisfies $V \cap j(X) \subseteq j(U)$ the Hahn-Banach theorem gives $\tilde{f}_i \in V^\circ$ with $\tilde{f}_i \circ j = f_i$. Then $\tilde{f}(y) := (\tilde{f}_i(y))_{i\in M}$ defines an extension $\tilde{f} : Y \longrightarrow \ell_M^\infty$ of $f$.

Next, we observe that every l.c.s. endowed with the coarsest topology is injective and that the product of injective objects in $\mathcal{LCS}$ is again injective.

If now $X$ is any l.c.s. we endow $\overline{\{0\}}^X$ with the coarsest topology. Let $Y = X/\overline{\{0\}}$ be the associated separated space of $X$. Given $U \in \mathscr{U}_0(Y)$, there is a canonical map $p_U : Y \longrightarrow \ell^\infty_{U^\circ},\ y \mapsto (\varphi(y))_{\varphi \in U^\circ}$ which is well-defined (since $U$ is absorbant) and obviously continuous. Since $Y$ is separated we obtain a monohomomorphism $p : Y \longrightarrow \prod_{U \in \mathscr{U}_0(Y)} \ell^\infty_{U^\circ},\ y \mapsto (p_U(y))_{u \in \mathscr{U}_0(Y)}$, and finally a monohomomorphism $X \longrightarrow \overline{\{0\}}^X \times \prod_{U \in \mathscr{U}_0(Y)} \ell^\infty_{U^\circ}$. □

Form the above proof and the fact that separated quotients of Fréchet (Banach) spaces belong to the same class we obtain that every Fréchet (Banach) space has an injective resolution consisting of Fréchet (Banach) spaces. Note however that in the category $\mathscr{F}$ of Fréchet spaces cokernels of morphisms $f : X \to Y$ are different from those in $\mathcal{LCS}$. The cokernel in $\mathscr{F}$ is the quotient map $Y \longrightarrow Y/\overline{f(X)}$. This difference is not visible as long as one considers exact complexes. A complex

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is exact in $\mathscr{F}$ iff it is exact in $\mathcal{LCS}$ iff it is exact in the category of vector spaces (the latter condition is called algebraic exactness), which follows from the open mapping theorem.

Contrary to the existence of injective objects in $\mathcal{LCS}$ there are only very few projective objects. Answering problem [50, §12.1] of Palamodov, V.A. Geĭler [30] proved that direct sums of the field are the only projective objects. This contrasts the simple fact that the space $\ell^1$ of absolutely summable sequences is projective in the category of Banach spaces.

In the category of locally convex spaces, a complex

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is acyclic iff it is algebraically exact. To measure the “lack of openness” of $f$ and $g$, Palamodov introduced contravariant functors $H_M$ (where $M$ is any set) from $\mathcal{LCS}$ to the category of vector spaces: $H_M(X) = \mathrm{Hom}(X, \ell^\infty_M)$ where $\ell^\infty_M$ is the Banach space defined in the proof of 2.2.1 and $f : X \longrightarrow Y$ is transformed into $f^* : H_M(Y) \longrightarrow H_M(X), T \mapsto T \circ f$. The value of these functors comes from the following result.

**Theorem 2.2.2** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a complex in $\mathcal{LCS}$. The following conditions are equivalent.*

1. $\mathrm{im}\, f$ *is dense in* $\ker g$ *and* $g$ *is a homomorphism.*
2. *For each set $M$ the complex*

$$H_M(Z) \xrightarrow{g^*} H_M(Y) \xrightarrow{f^*} H_M(X)$$

*is exact (in the category of vector spaces).*

3. *the complex in 2. is exact for some set $M$ whose cardinality is larger than that of* $(\mathrm{coim}\, g)'$.

**Proof.** If the first condition is satisfied we consider for some set $M$ the complex in 2. Let $S \in H_M(Y)$ be given with $f^*(S) = S \circ f = 0$. We denote by $S_i$ the composition of $S$ with the projection onto the $i$-the component of $\ell^\infty_M$. Since $S$ is continuous there is $U \in \mathscr{U}_0(Y)$ with $S_i \in U^\circ$ for all $i \in M$. Moreover, $S_i \in (\text{im f})^\circ = (\overline{\text{im f}})^\circ = (\text{ker g})^\circ$, hence

$$t_i : \text{im g} \longrightarrow \mathbb{K},\ g(y) \longmapsto S_i(y)$$

is a well defined linear map. If $V \in \mathscr{U}_0(Z)$ satisfies $V \cap \text{im g} \subseteq g(U)$ we obtain $t_i \in (\text{im g} \cap V)^\circ$ and we can use the Hahn-Banach theorem to find extensions $T_i \in V^\circ$ with $T_i|_{\text{im g}} = t_i$. Then $T(z) = (T_i(z))_{i \in M}$ defines an element of $H_M(Z)$ with $g^*(T) = T \circ g = S$.

If the third condition holds we first note that the complexes in 2 are exact for all smaller sets. Taking a set $M$ with one element we obtain that

$$Z' \xrightarrow{g^t} Y' \xrightarrow{f^t} X'$$

is (algebraically) exact which is equivalent to $\text{im}\, f$ being dense in $\ker g$ again by the Hahn-Banach theorem.

Let now $U = \overline{U} \in \mathscr{U}_0(Y)$ be given and set $M = (U + \text{im}\, f)^\circ = (U + \ker g)^\circ$ which can be identified with a subset of $(\text{coim}\, g)'$. Then

$$S(y) = (S_\varphi(y))_{\varphi \in M} = (\varphi(y))_{\varphi \in M}$$

defines an element of $H_M(Y)$ with $f^*(S) = 0$. Hence there is $T = (T_\varphi)_{\varphi \in M} \in H_M(Z)$ with $g^*(T) = S$. Choosing $V \in \mathscr{U}_0(Z)$ with $T_\varphi \in V^\circ$ for all $\varphi$ we obtain for $\varphi \in M$ and $y \in g^{-1}(V)$ that $|\varphi(y)| = |S_\varphi(y)| = |T_\varphi(g(y))| \leq 1$, i.e. $(U + \text{im}\, f)^\circ \subseteq g^{-1}(V)^\circ$ and thus $g^{-1}(V) \subseteq (U + \text{im}\, f)^{\circ\circ} \subseteq 2U + \text{im}\, f$. This yields $V \cap \text{im g} \subseteq g(2U + \text{im f}) = g(2U)$, hence $g$ is open onto its range. □

# 3

# The projective limit functor for countable spectra

This chapter is the core these notes. The characterizations of $\operatorname{Proj}^1 \mathscr{X} = 0$ obtained in 3.2 and 3.3 have many applications in analysis (some of them are presented in 3.4) and are used to study other functors in chapters 5, 6, and 7.

## 3.1 Projective limits of linear spaces

The way we introduce countable spectra and the projective limit functor differs from Palamodov's definition [49, 50] which has certain advantages but is a bit technical. Our naive approach is very simple but on the other hand it requires some arrangements in applications which are explained below.

**Definition 3.1.1** *A projective spectrum $\mathscr{X}$ is a sequence $(X_n)_{n\in\mathbb{N}}$ of linear spaces (over the same scalar field) together with linear maps $\varrho_m^n : X_m \longrightarrow X_n$ for $n \le m$ such that $\varrho_n^n = id_{X_n}$ and $\varrho_n^k \circ \varrho_m^n = \varrho_m^k$ for $k \le n \le m$. For two spectra $\mathscr{X} = (X_n, \varrho_m^n)$ and $\mathscr{Y} = (Y_n, \sigma_m^n)$ a morphism $f = (f_n)_{n\in\mathbb{N}} : \mathscr{X} \longrightarrow \mathscr{Y}$ consists of linear maps $f_n : X_n \longrightarrow Y_n$ such that $f_n \circ \varrho_m^n = \sigma_m^n \circ f_m$ for $n \le m$, i.e. the diagram*

$$\begin{array}{ccc} X_n & \xrightarrow{\;f_n\;} & Y_n \\ {\scriptstyle \varrho_m^n}\big\uparrow & & \big\uparrow{\scriptstyle \sigma_m^n} \\ X_m & \xrightarrow{\;f_m\;} & Y_m \end{array}$$

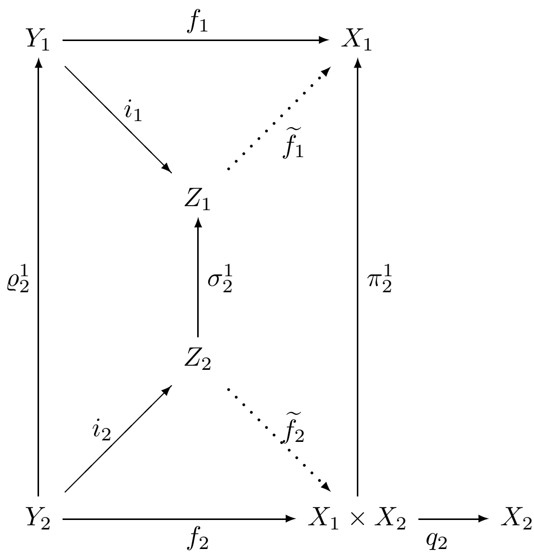
*is commutative. The sum and composition of two morphisms are defined in the obvious way by adding and composing the components of the morphisms, respectively.*

**Proposition 3.1.2** *The class of projective spectra and morphisms forms an abelian category which has sufficiently many injective objects.*

J. Wengenroth: LNM 1810, pp. 17–57, 2003.

**Proof.** It is very easy to see that the category is additive and that for a morphism $f = (f_n)_{n\in\mathbb{N}} : \mathscr{X} \longrightarrow \mathscr{Y}$ we have a kernel $i = (i_n)_{n\in\mathbb{N}} : (\ker f_n)_{n\in\mathbb{N}} \longrightarrow \mathscr{X}$, where $i_n : \ker f_n \longrightarrow X_n$ is the kernel of $f_n$, and a cokernel $q = (q_n)_n : \mathscr{Y} \longrightarrow (\operatorname{coker} f_n)_{n\in\mathbb{N}}$, where $q_n : Y_n \longrightarrow \operatorname{coker} f_n$ is the cokernel of $f_n$. Moreover, it is immediate that every morphism is a homomorphism since this is so in the category of linear spaces.

Let us show that for every sequence $(X_n)_{n\in\mathbb{N}}$ of linear spaces the corresponding free spectrum $\mathscr{I} = (I_n, \pi^n_m)$ is injective, where $I_n = \prod_{k=1}^{n} X_k$ and $\pi^n_m : I_m \longrightarrow I_n$ is the canonical projection. Let $\mathscr{Y} = (Y_n, \varrho^n_m)$ and $\mathscr{Z} = (Z_n, \sigma^n_m)$ be two spectra, $f = (f_n)_{n\in\mathbb{N}} : \mathscr{Y} \longrightarrow \mathscr{I}$ a morphism and $i = (i_n)_{n\in\mathbb{N}} : \mathscr{Y} \longrightarrow \mathscr{Z}$ a monomorphism. We have the following commutative diagram (in which we first ignore the dotted arrows)




where $q_2 : X_1 \times X_2 \longrightarrow X_2$ is the projection. Let $\widetilde{f}_1 : Z_1 \longrightarrow X_1$ be an extension of $f_1$ and $\widetilde{f}^2_2$ an extension of $f^2_2 = q_2 \circ f_2$. Defining

$$\widetilde{f}_2 = (\widetilde{f}_1 \circ \sigma^1_2, \widetilde{f}^2_2) : Z_2 \longrightarrow X_1 \times X_2$$

we get an extension of $f_2$ such that the extended diagram remains commutative. Continuing in that way we construct an extension $\widetilde{f} = (\widetilde{f}_n)_{n\in\mathbb{N}} : \mathscr{Z} \longrightarrow \mathscr{I}$ of $f$. If finally $\mathscr{X} = (X_n, \varrho^n_m)$ is any spectrum and $\mathscr{I}$ is the corresponding free spectrum we have a monomorphism $i = (i_n)_{n\in\mathbb{N}} : \mathscr{X} \longrightarrow \mathscr{I}$, where $i_n = (\varrho^1_n, \varrho^2_n, \ldots, \varrho^n_n)$. □

**Definition 3.1.3** *For a spectrum $\mathscr{X} = (X_n, \varrho^n_m)$ we set*

$$\operatorname{Proj} \mathscr{X} = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \varrho^n_m(x_m) = x_n \text{ for all } n \leq m \right\},$$

*and for a morphism* $f = (f_n)_{n\in\mathbb{N}} : \mathscr{X} \longrightarrow \mathscr{Y}$ *we define*

$$\operatorname{Proj} f : \operatorname{Proj}\mathscr{X} \longrightarrow \operatorname{Proj}\mathscr{Y} \text{ by } (x_n)_{n\in\mathbb{N}} \mapsto (f_n x_n)_{n\in\mathbb{N}}.$$

*Moreover,* $\varrho^n : \operatorname{Proj}\mathscr{X} \longrightarrow X_n$ *denotes the projection onto the* $n$*-th component.*

Note that $\operatorname{Proj} f$ is a well-defined linear map since $f$ is a morphism, and that $\operatorname{Proj}\mathscr{X}$ is the kernel of the map

$$\Psi = \Psi_{\mathscr{X}} : \prod_{n\in\mathbb{N}} X_n \longrightarrow \prod_{n\in\mathbb{N}} X_n,\ (x_n)_n \mapsto (x_n - \varrho^n_{n+1} x_{n+1})_{n\in\mathbb{N}}.$$

With these definitions, Proj is a functor acting on the category of projective spectra with values in the category of linear spaces, and this functor is easily seen to be injective. Since the category of projective spectra has many injective objects the derived functors $\operatorname{Proj}^k\mathscr{X}$ can be defined by the homological construction described in chapter 2. Using the injective objects constructed in the proof of 3.1.2 we find an explicit description of $\operatorname{Proj}^k\mathscr{X}$.

**Theorem 3.1.4** *Let* $\mathscr{X} = (x_n, \varrho^n_m)$ *be a projective spectrum. Then*

$$\operatorname{Proj}^0\mathscr{X} = \operatorname{Proj}\mathscr{X},\ \operatorname{Proj}^k\mathscr{X} = 0 \text{ for } k \geq 2, \text{ and}$$

$$\operatorname{Proj}^1\mathscr{X} \cong \left(\prod_{n\in\mathbb{N}} X_n\right) / \operatorname{im}\Psi,$$

*where* $\Psi\left((x_n)_n\right) = \left(x_n - \varrho^n_{n+1}(x_{n+1})\right)_{n\in\mathbb{N}}$.

**Proof.** Let $\mathscr{I} = (\prod_{j=1}^n X_j, \pi^n_m)$ be the free spectrum corresponding to the sequence $(X_n)_{n\in\mathbb{N}}$ and $i = (i_n)_{n\in\mathbb{N}} : \mathscr{X} \longrightarrow \mathscr{I}$ the monomorphism defined by $i_n = (\varrho^1_n, \ldots, \varrho^n_n)$ as in the proof of 3.1.2. We set $\mathscr{K} = (\prod_{j=1}^{n-1} X_j, q^n_m)$, where $q^n_m$ are again the canonical projections, and $k = (k_n)_{n\in\mathbb{N}} : \mathscr{I} \longrightarrow \mathscr{K}$, where $k_n(x_1, \ldots, x_n) = (x_1 - \varrho^1_2 x_2, \ldots, x_{n-1} - \varrho^{n-1}_n x_n)$. Then $k$ is an epimorphism and

$$0 \longrightarrow \mathscr{X} \xrightarrow{\ i\ } \mathscr{I} \xrightarrow{\ k\ } \mathscr{K} \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

is an injective resolution of $\mathscr{X}$. We therefore obtain $\operatorname{Proj}^0\mathscr{X} = \ker(\operatorname{Proj} k)$, $\operatorname{Proj}^1\mathscr{X} = \operatorname{Proj}\mathscr{K}/\operatorname{im}(\operatorname{Proj} k)$, and $\operatorname{Proj}^m\mathscr{X} = 0$ for $m \geq 2$. On the other hand,

$$T : \operatorname{Proj}\mathscr{I} \longrightarrow \prod_{n\in\mathbb{N}} X_n,\ (x_{jn})_{j\leq n} \mapsto (x_{nn})_{n\in\mathbb{N}}$$

is an isomorphism which maps $\ker(\operatorname{Proj} k)$ onto $\operatorname{Proj}\mathscr{X}$, and

$$S : \operatorname{Proj}\mathscr{K} \longrightarrow \prod_{n\in\mathbb{N}} X_n,\ (x_{jn})_{j<n} \mapsto (x_{n,n+1})_{n\in\mathbb{N}}$$

is an isomorphism which maps the image of $\operatorname{Proj} k$ onto the image of $\Psi$. □

It is immediate from the definition that a short exact sequence

$$0 \longrightarrow \mathscr{X} \longrightarrow \mathscr{Y} \longrightarrow \mathscr{Z} \longrightarrow 0$$

of projective spectra consists of short exact sequences of linear spaces such that the following diagrams are commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_n & \longrightarrow & Y_n & \longrightarrow & Z_n & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & X_m & \longrightarrow & Y_m & \longrightarrow & Z_m & \longrightarrow & 0. \end{array}$$

The next corollary is clear from theorem 2.1.1, and it indicates the typical situation of applications. If one wants to avoid the homological tools and defines instead $\operatorname{Proj}^1 \mathscr{X}$ by the formula in 3.1.4 one can easily give a direct proof by diagram chasing.

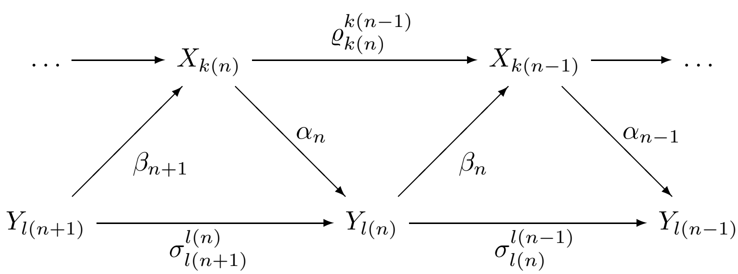
**Corollary 3.1.5** *Let* $0 \longrightarrow \mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z} \longrightarrow 0$ *be a short exact sequence of projective spectra. Then there is an exact sequence*

$$0 \longrightarrow \operatorname{Proj} \mathscr{X} \longrightarrow \operatorname{Proj} \mathscr{Y} \longrightarrow \operatorname{Proj} \mathscr{Z}$$

$$\xrightarrow{\delta} \operatorname{Proj}^1 \mathscr{X} \longrightarrow \operatorname{Proj}^1 \mathscr{Y} \longrightarrow \operatorname{Proj}^1 \mathscr{Z} \longrightarrow 0.$$

To explain the difference between Palamodov's terminology and the way we defined projective spectra, we use:

**Definition 3.1.6** *Two spectra* $\mathscr{X} = (X_n, \varrho^n_m)$ *and* $\mathscr{Y} = (Y_n, \sigma^n_m)$ *are equivalent if there are increasing sequences* $(k(n))_{n\in\mathbb{N}}$ *and* $(l(n))_{n\in\mathbb{N}}$ *of natural numbers with* $n \leq l(n) \leq k(n) \leq l(n+1)$ *and linear maps* $\alpha_n : X_{k(n)} \longrightarrow Y_{l(n)}$, $\beta_n : Y_{l(n)} \longrightarrow X_{k(n-1)}$ *such that* $\beta_n \circ \alpha_n = \varrho^{k(n-1)}_{k(n)}$ *and* $\alpha_n \circ \beta_{n+1} = \sigma^{l(n)}_{l(n+1)}$, *i.e. the following diagram commutes:*




Palamodov defined projective spectra as classes of equivalent spectra. The advantage of this approach is that in many applications the projective limits are the given objects and there are different ways to construct projective spectra with that limit. To apply the theory we developed here one has to choose spectra and morphisms carefully to get an exact sequence of projective spectra.

On the other hand there may be different spectra having a concrete meaning and thus allow calculations required in the results presented in sections 3.2 and 3.3, but which do not yield exact sequences of spectra in our sense. We will now show that $\operatorname{Proj}^k \mathscr{X}$ is invariant under equivalent spectra (this comes out of Palamodov's theory directly) which allows us to use one spectrum to which our notion of exact sequences applies and a perhaps different but equivalent one to calculate $\operatorname{Proj}^1 \mathscr{X}$.

**Proposition 3.1.7** *If $\mathscr{X}$ and $\mathscr{Y}$ are equivalent projective spectra then we have* $\operatorname{Proj}^k \mathscr{X} \cong \operatorname{Proj}^k \mathscr{Y}$ *for all $k \in \mathbb{N}_0$.*

**Proof.** We first consider the special case where $\mathscr{Y}$ is a "subsequence" of $\mathscr{X}$, i.e. for some strictly increasing sequence $(k(n))_{n\in\mathbb{N}}$ of natural numbers we have $Y_n = X_{k(n)}$, $\sigma^n_{n+1} = \varrho^{k(n)}_{k(n+1)}$, $\alpha_n = \varrho^n_{k(n)}$, and $\beta_n = id_{X_{k(n)}}$. Then $\operatorname{Proj} \mathscr{X} \cong \operatorname{Proj} \mathscr{Y}$ is obvious. We will show, that

$$T : \prod_{n\in\mathbb{N}} X_n \longrightarrow \prod_{n\in\mathbb{N}} X_{k(n)},\ (x_n)_{n\in\mathbb{N}} \mapsto \left( \sum_{j=k(n)}^{k(n+1)-1} \varrho_j^{k(n)} x_j \right)_{n\in\mathbb{N}}$$

induces an isomorphism between $\operatorname{Proj}^1 \mathscr{X}$ and $\operatorname{Proj}^1 \mathscr{Y}$. If $x = (x_n)_{n\in\mathbb{N}} = (z_n - \varrho^n_{n+1} z_{n+1})_{n\in\mathbb{N}} \in \operatorname{im}(\Psi_{\mathscr{X}})$ then $T(x) \in \operatorname{im}(\Psi_{\mathscr{Y}})$, since for each $n \in \mathbb{N}$

$$\begin{aligned}\sum_{j=k(n)}^{k(n+1)-1} \varrho_j^{k(n)} (z_j - \varrho^j_{j+1} z_{j+1}) &= \sum_{j=k(n)}^{k(n+1)-1} \varrho_j^{k(n)} z_j - \sum_{j=k(n)+1}^{k(n+1)} \varrho_j^{k(n)} z_j \\ &= z_{k(n)} - \varrho^{k(n)}_{k(n+1)} z_{k(n+1)}.\end{aligned}$$

This shows that $T$ induces a linear map $\hat{T} : \operatorname{Proj}^1 \mathscr{X} \longrightarrow \operatorname{Proj}^1 \mathscr{Y}$, which is surjective since $T$ is surjective.

To prove that $\hat{T}$ is injective let $x = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ be given with $T(x) = \left( y_{k(n)} - \varrho^{k(n)}_{k(n+1)} y_{k(n+1)} \right)_{n\in\mathbb{N}} \in \operatorname{im}(\Psi_{\mathscr{Y}})$. For $r \in \mathbb{N}$ with $k(n) \le r < k(n+1)$ we set $z_r := \varrho^r_{k(n+1)} y_{k(n+1)} + \sum_{j=r}^{k(n+1)-1} \varrho^r_j x_j$. Then $z_r - \varrho^r_{r+1} z_{r+1} = x_r$ which is clear for $k(n) \le r < k(n+1)-1$, and it also holds for $r = k(n+1)-1$ because of

$$\begin{aligned}\varrho^r_{r+1} z_{r+1} &= \varrho^r_{r+1} \left( \sum_{j=k(n+1)}^{k(n+2)-1} \varrho_j^{k(n+1)} x_j + \varrho^{k(n+1)}_{k(n+2)} y_{k(n+2)} \right) \\ &= \varrho^r_{r+1} y_{k(n+1)} = z_r - x_r.\end{aligned}$$

Therefore, $x \in \operatorname{im}(\Psi_{\mathscr{X}})$, and this shows that $\hat{T}$ is injective, hence an isomorphism.

By passing to subsequences of $\mathscr{X}$ and $\mathscr{Y}$ we can now reduce the general case to the particular one where we have linear maps $\alpha_n : X_n \longrightarrow Y_n$ and $\beta_{n+1} : Y_{n+1} \longrightarrow X_n$ with $\alpha_n \circ \beta_{n+1} = \sigma^n_{n+1}$ and $\beta_n \circ \alpha_n = \varrho^{n-1}_n$. Defining

$$S : \prod_{n\in\mathbb{N}} X_n \longrightarrow \prod_{n\in\mathbb{N}} Y_n,\ (x_n)_{n\in\mathbb{N}} \mapsto (\alpha_n x_n)_{n\in\mathbb{N}} \text{ and}$$
$$R : \prod_{n\in\mathbb{N}} Y_n \longrightarrow \prod_{n\in\mathbb{N}} X_n,\ (y_n)_{n\in\mathbb{N}} \mapsto (\beta_{n+1} y_{n+1})_{n\in\mathbb{N}}$$

we compute $(S\circ R)((y_n)_{n\in\mathbb{N}}) = (\sigma^n_{n+1} y_{n+1})_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}} - \Psi_{\mathscr{Y}}((y_n)_{n\in\mathbb{N}})$ and $(R \circ S)((x_n)_{n\in\mathbb{N}}) = (\varrho^n_{n+1} x_{n+1})_{n\in\mathbb{N}} = (x_n)_{n\in\mathbb{N}} - \Psi_{\mathscr{X}}((x_n)_{n\in\mathbb{N}})$. This shows that $S$ induces an isomorphism $\widetilde{S} : \operatorname{Proj} \mathscr{X} \longrightarrow \operatorname{Proj} \mathscr{Y}$ and an isomorphism $\hat{S} : \operatorname{Proj}^1 \mathscr{X} \longrightarrow \operatorname{Proj}^1 \mathscr{Y}$. □

When one investigates the surjectivity of a map $g : Y \to Z$ using *canonical* projective descriptions $Y = \operatorname{Proj} \mathscr{Y}$ and $Z = \operatorname{Proj} \mathscr{Z}$ it may happen that $\mathscr{Y} \to \mathscr{Z}$ is not an epimorphism in our category (i.e. $Z_n \neq g_n(Y_n)$) but we only have $\tau^n_m(Z_m) \subseteq g_n(Y_n)$ for some $m = m(n)$. This is the case in most of the examples given in section 3.4 below. Instead of rearranging the spectra to get an epimorphism it is more convenient to apply the following proposition which gives surjectivity of $g$ if the kernel spectrum $\mathscr{X}$ satisfies $\operatorname{Proj}^1 \mathscr{X} = 0$.

**Proposition 3.1.8** *Let $g = (g_n)_{n\in\mathbb{N}} : \mathscr{Y} \longrightarrow \mathscr{Z}$ be a morphism between two spectra $\mathscr{Y} = (Y_n, \sigma^n_m)$ and $\mathscr{Z} = (Z_n, \tau^n_m)$ such that*

$$\forall\, n \in \mathbb{N} \quad \exists\, m(n) \geq n \quad \tau^n_{m(n)}(Z_{m(n)}) \subseteq g_n(Y_n).$$

*If $\mathscr{X}$ is the kernel of $g$ there is an exact complex*

$$0 \to \operatorname{Proj} \mathscr{X} \to \operatorname{Proj} \mathscr{Y} \to \operatorname{Proj} \mathscr{Z} \to \operatorname{Proj}^1 \mathscr{X} \to \operatorname{Proj}^1 \mathscr{Y} \to \operatorname{Proj}^1 \mathscr{Z} \to 0.$$

**Proof.** With $k(1) = 1$ and $k(n+1) = m(k(n))$ we get spectra $\widetilde{\mathscr{Y}} = (Y_{k(n)}, \sigma^{k(n)}_{k(m)})$ and $\widetilde{\mathscr{Z}} = (g_{k(n)}(Y_{k(n)}), \tau^{k(n)}_{k(m)})$ which are equivalent to $\mathscr{Y}$ and $\mathscr{Z}$ respectively, and an *epimorphism* $\widetilde{g} = (g_{k(n)}) : \widetilde{\mathscr{Y}} \longrightarrow \widetilde{\mathscr{Z}}$ such that

$$\begin{array}{ccc} \operatorname{Proj} \mathscr{Y} & \xrightarrow{\operatorname{Proj} g} & \operatorname{Proj} \mathscr{Z} \\ \downarrow & & \uparrow \\ \operatorname{Proj} \widetilde{\mathscr{Y}} & \xrightarrow{\operatorname{Proj} \widetilde{g}} & \operatorname{Proj} \widetilde{\mathscr{Z}} \end{array}$$

is commutative (where the vertical arrows are the canonical isomorphisms). Moreover, the kernels of $g$ and $\widetilde{g}$ are equivalent spectra and we thus get the assertion from 3.1.5 and 3.1.7. □

In Palamodov's approach, morphisms are equivalence classes of morphisms in our sense, and an epimorphism is then characterized by having a representative satisfying the condition of the proposition above. The scope of possible applications of both theories is therefore the same.

## 3.2 The Mittag-Leffler procedure

We will now present Palamodov's [49] sufficient condition for a spectrum $\mathscr{X}$ to satisfy $\operatorname{Proj}^1 \mathscr{X} = 0$. This happens if there are complete metrizable group topologies on the steps such that the linking maps become continuous with dense range. We will present three proofs of this result. The standard proof where the surjectivity of the map $\Psi$ is achieved by writing down solutions as convergent series, a second one which reduces the result to the classical abstract Mittag-Leffler lemma, and a third one using the Schauder lemma.

**Theorem 3.2.1** *Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a projective spectrum and assume that each $X_n$ is endowed with a complete metrizable group topology such that the spectral maps are continuous and*

$$\forall\, n \in \mathbb{N},\, U \in \mathscr{U}_0(X_n)\ \exists\, m \geq n\ \forall\, k \geq m \quad \varrho_m^n X_m \subseteq \varrho_k^n X_k + U.$$

*Then* $\operatorname{Proj}^1 \mathscr{X} = 0$.

**First Proof.** Let $(U_{n,N})_{N\in\mathbb{N}}$ be bases of $\mathscr{U}_0(X_n)$ such that

$$U_{n,N+1} + U_{n,N+1} \subseteq U_{n,N} \text{ and } \varrho_m^n(U_{m,N}) \subseteq U_{n,N}.$$

Such bases exist because of the continuity of $+$ on $X_n$ and of $\varrho_m^n$. Since it is by 3.1.7 enough to show $\operatorname{Proj}^1 \widetilde{\mathscr{X}} = 0$ for a subsequence $\widetilde{\mathscr{X}}$ we may assume

$$\varrho_{n+1}^n X_{n+1} \subseteq \varrho_{n+2}^n X_{n+2} + U_{n,n} \text{ for all } n \in \mathbb{N}.$$

Given $x = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ we set $y_1 = y_2 = 0$ and choose inductively $y_{n+2} \in X_{n+2}$ and $z_n \in U_{n,n}$ such that $\varrho_{n+1}^n(y_{n+1} - x_{n+1}) = \varrho_{n+2}^n(y_{n+2}) + z_n$. The arrangement of the neighbourhoods and the completeness of $X_n$ imply that $r_n = \sum_{k=n}^{\infty} \varrho_k^n(z_k)$ converges in $X_n$, and the continuity of $\varrho_{n+1}^n$ gives

$$r_n - \varrho_{n+1}^n(r_{n+1}) = z_n.$$

Now we define $w_n = x_n + \varrho_{n+1}^n(y_{n+1}) - r_n$ and obtain a solution $w = (w_n)_{n\in\mathbb{N}}$ of $\Psi(w) = x$ since

$$\begin{aligned}\varrho_{n+1}^n(w_{n+1}) &= \varrho_{n+1}^n(x_{n+1}) + \varrho_{n+2}^n(y_{n+2}) - \varrho_{n+1}^n(r_{n+1})\\ &= \varrho_{n+1}^n(y_{n+1}) - z_n - \varrho_{n+1}^n(r_{n+1})\\ &= \varrho_{n+1}^n(y_{n+1}) - r_n = w_n - x_n.\end{aligned}$$

□

The second (of course similar) proof uses the following version of the classical Mittag-Leffler lemma for projective spectra of complete metric spaces (i.e. the objects $X_n$ in definition 3.1.1 are complete metric spaces and the spectral maps $\varrho_m^n$ are assumed to be continuous maps).

For a subset $A$ of a metric space $(X, d)$ we denote by $U_\varepsilon(A)$ the neighbourhood $\{x \in X : \exists\, a \in A \quad d(x, a) < \varepsilon\}$.

The proof of the following lemma is the same as e.g. in [12, chapter 2, §3].

**Lemma 3.2.2** *Let $\mathscr{X} = (X_n, \sigma_m^n)$ be a projective spectrum of complete metric spaces such that*

$$\forall\, n \in \mathbb{N},\, \varepsilon > 0\ \exists\, m \geq n\ \forall\, k \geq m \quad \sigma_m^n(X_m) \subseteq U_\varepsilon(\varrho_k^n(X_k)).$$

$$\textit{Then}\ \forall\, n \in \mathbb{N},\, \varepsilon > 0\ \exists\, m \geq n \quad \sigma_m^n(X_m) \subseteq U_\varepsilon(\varrho^n(\mathrm{Proj}\, \mathscr{X})).$$

**Second proof of 3.2.1.** Given $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ we define

$$\sigma_{n+1}^n(y) = \varrho_{n+1}^n(y) + x_n \text{ and } \sigma_m^n = \sigma_{n+1}^n \circ \ldots \circ \sigma_m^{m-1}.$$

There are complete invariant metrics $d_n$ on $X_n$ inducing the group topology on $X_n$, and the hypotheses of 3.2.1 imply those of 3.2.2 which yields that

$$\mathrm{Proj}\,(X_n, \sigma_m^n) = \left\{ (y_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \sigma_{n+1}^n(y_{n+1}) = y_n \right\}$$

is not empty, and each of its elements solves $y_n - \varrho_{n+1}^n(y_{n+1}) = x_n$. □

The preceding proof suggests the interpretation that $\mathrm{Proj}^1 \mathscr{X} = 0$ holds if and only if for every $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ the perturbed spectrum $\widetilde{\mathscr{X}} = (X_n, \sigma_m^n)$ where $\sigma_{n+1}^n(y) = y + x_n$ has non-empty projective limit. This viewpoint was promoted by G.R. Allan (without mentioning the derived functors of Proj) in a series of articles [1, 2, 3], where he calls a spectrum $\mathscr{X}$ with $\mathrm{Proj}^1 \mathscr{X} = 0$ a stable inverse limit sequence.

Our third proof of 3.2.1 uses the Schauder lemma for complete metric spaces, see e.g. [45, lemma 3.9]. We call a map $f : X \longrightarrow Y$ between metric spaces uniformly almost open if for each $\varepsilon > 0$ there is $\delta > 0$ such that $U_\delta\,(f\,(x)) \subseteq \overline{f\,(U_\varepsilon\,(x))}$ for every $x \in X$.

If $X$ and $Y$ are metric groups and $f$ is a group homomorphism, then $f$ is uniformly almost open if and only if $\overline{f(U)}$ is a neighbourhood of the unit element in $Y$ for each neighbourhood $U$ of the unit element in $X$.

**Schauder Lemma 3.2.3** *Let $X$ and $Y$ be metric spaces such that $X$ is complete. If a map from $X$ to $Y$ is uniformly almost open and has closed graph then it is open.*

To apply this and for later purposes we need:

**Lemma 3.2.4** *Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a projective spectrum, $A_j \subset X_j$ arbitrary subsets and $n \leq m \leq k$.*

1. *$\varrho_m^n X_m \subseteq \varrho_k^n X_k + \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(A_j)$ holds if and only if*

$$\prod_{j<m} \{0\} \times \prod_{j\geq m} X_j \subseteq \Psi_{\mathscr{X}}(\prod_{j\leq n} A_j \times \prod_{j>n} X_j) + \prod_{j<k} \{0\} \times \prod_{j\geq k} X_j.$$

*2.* $\prod_{j<m}\{0\} \times \prod_{j\geq m} X_j \subseteq \Psi_{\mathscr{X}}(\prod_{j\leq n} A_j \times \prod_{j>n} X_j)$ *implies*

$$\varrho^n_m X_m \subseteq \varrho^n(\operatorname{Proj}\mathscr{X}) + \bigcap_{j=1}^{n} (\varrho^j_n)^{-1}(A_j).$$

**Proof.** Suppose that the second condition of 1. holds and let $x_m \in X_m$ be given. Let $x$ be the sequence with all components 0 but the $m^{th}$ which is $x_m$. Then there is $y = (y_n)_{n\in\mathbb{N}} \in \prod_{j\leq n} A_j \times \prod_{j>n} X_j$ with

$$x - \Psi(y) \in \prod_{j<k}\{0\} \times \prod_{j\geq k} X_j.$$

Hence $y_j = \varrho^j_{y+1}(y_{j+1})$ for $1 \leq j < k$, $j \neq m$ and $x_m = y_m - \varrho^m_{m+1}(y_{m+1})$. This gives

$$\begin{aligned}\varrho^n_m(x_m) &= \varrho^n_m(y_m) - \varrho^n_{m+1}(y_{m+1}) = y_n - \varrho^n_k(y_k)\\ &\in \bigcap_{j=1}^{n}(\varrho^j_n)^{-1}(A_j) + \varrho^n_k(X_k).\end{aligned}$$

If the first condition of 1. is satisfied and $x = (x_j)_{j\in\mathbb{N}}$ belongs to $\prod_{j<m}\{0\} \times \prod_{j\geq m} X_j$ we set $t = \sum_{j=m}^{k-1} \varrho^m_j x_j$ and find $y \in \bigcap_{j=1}^{n} (\varrho^j_n)^{-1}(A_j)$ and $z \in X_k$ with $\varrho^n_m(t) = y + \varrho^n_k(z)$. With $v = t - \varrho^m_k(z) \in X_m$ we have $\varrho^n_m(v) = y$ and $x_m = v + \varrho^m_k(z) - \sum_{j=m+1}^{k-1} \varrho^m_j(x_j)$. With $u_j = \sum_{l=j}^{k-1} \varrho^j_l(x_l) - \varrho^j_k(z)$ this gives

$$\begin{aligned}&(0,\ldots,x_m,\ldots,x_{k-1},0,\ldots)\\ &= \Psi_{\mathscr{X}}(\varrho^1_m v,\ldots,\varrho^m_m v, u_{m+1},\ldots,u_{k-1},-z,0,\ldots) + (0,\ldots,0,z,0,\ldots)\\ &\in \Psi_{\mathscr{X}}(\prod_{j\leq n} A_j \times \prod_{j>n} X_j) + \prod_{j<k}\{0\} \times \prod_{j\geq k} X_j.\end{aligned}$$

Hence $x = (0,\ldots,x_m,\ldots,x_{k-1},0,\ldots) + (0,\ldots,x_k,x_{k+1},\ldots)$ also belongs to this set, since $\prod_{j<k}\{0\} \times \prod_{j\geq k} X_j$ is a linear space. The proof of the second part is similar but simpler. □

**Third proof of 3.2.1.** We set $X = \prod_{n\in\mathbb{N}}(X_n, \mathscr{T}_n)$ where $\mathscr{T}_n$ are the given group topologies on $X_n$ and $Y = \prod_{n\in\mathbb{N}}(X_n, \mathscr{S}_n)$ where $\mathscr{S}_n$ are the discrete topologies on $X_n$. $\Psi_{\mathscr{X}} : X \longrightarrow Y$ has closed graph since it is continuous as a map $X \longrightarrow X$ and the topology of $Y$ is finer than that of $X$. Since $\Psi_{\mathscr{X}}$ is additive and $X$ and $Y$ are topological groups $\Psi_{\mathscr{X}}$ will be uniformly almost open if we can show that the closure of $\Psi(U)$ belongs to $\mathscr{U}_0(Y)$ for each $U \in \mathcal{U}_0$ (X). Given $U \in \mathscr{U}_0(X)$ there are $n \in \mathbb{N}$ and $U_n \in \mathscr{U}_0(X_n, \mathscr{T}_n)$ such that

$$\prod_{j\leq n} \varrho^j_n(U_n) \times \prod_{j>n} X_j \subseteq U.$$

Choosing $m \geq n$ such that $\varrho^n_m X_m \subseteq \varrho^n_k X_k + U_n$ for all $k \geq m$ and applying the first part of lemma 3.2.4 we obtain

$$\prod_{j<m} \{0\} \times \prod_{j\geq m} X_j \subseteq \Psi(U) + \prod_{j<k} \{0\} \times \prod_{j\geq k} X_j \text{ for all } k \geq m.$$

The left hand side of this inclusion is a typical 0-neighbourhood of $Y$ and the intersection of all right hand sides is the closure of $\Psi(U)$ in $Y$.

Hence, $\Psi$ is uniformly almost open and thus open by the Schauder lemma, i.e. for all $U \in \mathscr{U}_0(X)$ there is $m \in \mathbb{N}$ such that $\prod_{j<m} \{0\} \times \prod_{j\geq m} X \subseteq \Psi_{\mathscr{X}}(U)$. Since $\prod_{j<m} X_j \times \prod_{j\geq m} \{0\} \subseteq \Psi_{\mathscr{X}}(X)$ we obtain that $\Psi_{\mathscr{X}}$ is surjective. □

Although this last proof looks more complicated than the others it has some advantages. It provides a setting in which one can show that in many cases a condition like in 3.2.1 is also necessary for $\operatorname{Proj}^1 \mathscr{X} = 0$, the quantifiers of theorem 3.2.1 appear quite naturally, and it immediately gives the following result (which nevertheless can also be proved with 3.2.2).

**Theorem 3.2.5** *Let $\mathscr{X} = (X_n, \varrho^n_m)$ be a projective spectrum consisting of complete metrizable topological groups and continuous maps such that*

$$\forall\, n \in \mathbb{N}, U \in \mathscr{U}_0(X_n)\ \exists\, m \geq n\ \forall\, k \geq n \quad \varrho^n_m X_m \subseteq \varrho^n_k X_k + U.$$

$$\textit{Then } \forall\, n \in \mathbb{N}, U \in \mathscr{U}_0(X_n)\ \exists\, m \geq n \quad \varrho^n_m X_m \subseteq \varrho^n(\operatorname{Proj} \mathscr{X}) + U.$$

**Proof.** In the third proof of 3.2.1 we have shown that $\Psi_{\mathscr{X}}$ is open. Together with the second part of 3.2.4 this gives the conclusion. □

We will now show the necessity of conditions like in theorem 3.2.1 for $\operatorname{Proj}^1 \mathscr{X} = 0$. The key to such results is the observation that in our third proof above $Y = \prod_{n\in\mathbb{N}} (X_n, \mathscr{S}_n)$ is a complete metric space, hence it has the Baire property.

**Proposition 3.2.6** *Let $\mathscr{X} = (X_n, \varrho^n_m)$ be a projective spectrum satisfying $\operatorname{Proj}^1 \mathscr{X} = 0$ and assume that each $X_n$ is the countable union of absolutely convex sets $A_{n,N}$. Then there is a sequence $(N(n))_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m \quad \varrho^n_m(X_m) \subseteq \varrho^n_k(X_k) + \bigcap_{j=1}^{n} (\varrho^j_n)^{-1}(A_{j,N(j)}).$$

**Proof.** We keep the notation of the third proof of theorem 3.2.1. Since $\Psi_{\mathscr{X}}$ is surjective we have

$$Y = \Psi_{\mathscr{X}}(\prod_{n\in\mathbb{N}} X_n) = \Psi_{\mathscr{X}}(\bigcup_{N\in\mathbb{N}} A_{1,N} \times \prod_{n\geq 2} X_n) = \bigcup_{N\in\mathbb{N}} \Psi_{\mathscr{X}}(A_{1,N} \times \prod_{n\geq 2} X_n).$$

Since $Y$ is a Baire space there is $N(1)$ such that $\Psi_{\mathscr{X}}(A_{1,N(1)} \times \prod_{n\geq 2} X_n)$ is not meager in $Y$. But this set equals $\Psi_{\mathscr{X}}(A_{1,N(1)} \times \bigcup_{N\in\mathbb{N}} A_{n,N} \times \prod_{n\geq 3} X_n) = \bigcup_{N\in\mathbb{N}} \Psi_{\mathscr{X}}(A_{1,N(1)} \times A_{2,N} \times \prod_{n\geq 3} X_n)$. Hence there is $N(2) \in \mathbb{N}$ such that

$$\Psi_{\mathscr{X}}(A_{1,N(1)} \times A_{2,N(2)} \times \prod_{n\geq 3} X_n)$$

is not meager. Inductively, we find $N(j) \in \mathbb{N}$ such that all the sets $B_n := \Psi_{\mathscr{X}}(\prod_{j\leq n} A_{j,N(j)} \times \prod_{j>n} X_j)$ are not meager in $Y$, hence their closures contain interior points. Since each $B_n$ is absolutely convex and there is a basis of $\mathscr{U}_0(Y)$ consisting of vector spaces one easily sees that 0 is in the interior of $\overline{B_n}$. Hence for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$\prod_{j<m} \{0\} \times \prod_{j\geq m} X_j \subseteq \overline{B_n} = \bigcap_{k\in\mathbb{N}} (B_n + \prod_{j<k} \{0\} \times \prod_{j\geq k} X_j).$$

Lemma 3.2.4 gives the conclusion. □

**Corollary 3.2.7** *Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a projective spectrum and $\mathscr{T}_n$ any locally convex topology on $X_n$. Then* $\operatorname{Proj}^1 \mathscr{X} = 0$ *implies*

$$\forall\, n \in \mathbb{N},\, U \in \mathscr{U}_0(X_n, \mathscr{T}_n)\ \exists\, m \geq n\ \forall\, k \geq m \quad \varrho_m^n(X_m) \subseteq \varrho_k^n(X_k) + U.$$

Combined with 3.2.1 we obtain the following characterization due to Palamodov [50, theorem 5.2].

**Theorem 3.2.8** *For a projective spectrum $\mathscr{X} = (X_n, \varrho_m^n)$ consisting of Fréchet spaces and continuous linear maps the following conditions are equivalent.*

1. $\operatorname{Proj}^1 \mathscr{X} = 0$.
2. $\forall\, n \in \mathbb{N},\, U \in \mathscr{U}_0(X_n)\ \exists\, m \geq n\ \forall\, k \geq m \quad \varrho_m^n X_m \subseteq \varrho_k^n X_k + U$.
3. $\forall\, n \in \mathbb{N},\, U \in \mathscr{U}_0(X_n)\ \exists\, m \geq n \quad \varrho_m^n X_m \subseteq \varrho^n(\operatorname{Proj} \mathscr{X}) + U$.

The next result is due to Retakh [54] and Palamodov [50, theorem 5.4]. Let us recall that a Banach disc is a bounded absolutely convex set which spans a Banach space.

**Theorem 3.2.9** *For a projective spectrum $\mathscr{X} = (X_n, \varrho_m^n)$ consisting of separated (LB)-spaces and continuous linear maps the following conditions are equivalent.*

1. $\operatorname{Proj}^1 \mathscr{X} = 0$.
2. *There is a sequence of Banach discs $B_n \subset X_n$ such that*

($\alpha$) $\varrho_m^n(B_m) \subseteq B_n$ *for* $n \leq m$,

($\beta$) $\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m \quad \varrho_m^n X_m \subseteq \varrho_k^n X_k + B_n$.

*3. There is a sequence of Banach discs $B_n \subset X_n$ such that*
($\alpha$) $\varrho_m^n(B_m) \subseteq B_n$ *for* $n \leq m$,
($\gamma$) $\forall\, n \in \mathbb{N}\ \exists\, m \geq n \quad \varrho_m^n X_m \subseteq \varrho^n(\mathrm{Proj}\,\mathscr{X}) + B_n$.

**Proof.** If $B_n \subset X_n$ are Banach discs satisfying ($\alpha$) and ($\beta$) the group topologies $\mathscr{T}_n$ having $\{\varepsilon B_n : \varepsilon > 0\}$ as bases of $\mathscr{U}_0(X_n, \mathscr{T}_n)$ satisfy the hypotheses of 3.2.1 and 3.2.5. These theorems give the equivalence of 2. and 3. and that these conditions imply 1. If $\mathrm{Proj}^1\mathscr{X} = 0$ and $(B_{n,N})_{N\in\mathbb{N}}$ are fundamental sequences of Banach discs in $X_n$, 3.2.6 implies the existence of a sequence $(N(n))_{n\in\mathbb{N}}$ of natural numbers such that $B_n := \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(B_{j,N(j)})$ satisfy $\alpha$) and $\beta$). Moreover, these sets are Banach discs which follows from the next lemma (remember that $\varrho_n^n = id_{X_n}$). □

**Lemma 3.2.10** *Let $f : X \longrightarrow Y$ be a continuous linear map between separated locally convex spaces and let $A \subset X$ and $B \subset Y$ be Banach discs. Then $f(A) + B$ and $A \cap f^{-1}(B)$ are again Banach discs.*

**Proof.** This follows from the fact that separated quotients and closed subspaces of Banach spaces are again Banach spaces and considering the exact sequence

$$0 \longrightarrow [A \cap f^{-1}(B)] \stackrel{i}{\longrightarrow} [A] \times [B] \stackrel{q}{\longrightarrow} [f(A) + B] \longrightarrow 0$$

where $i(x) = (x, -f(x))$ and $q(x, y) = f(x) + y$. $[f(A) + B]$ is separated since its unit ball $f(A) + B$ is bounded in $X$. □

Below, we will present two astonishing improvements of theorem 3.2.9 where the assumption ($\alpha$) which means that the spectral maps are continuous with respect to the group topologies $\mathscr{T}_n$ introduced in the proof of 3.2.9, can be removed. One could say that in this case a Mittag-Leffler procedure works without continuity or rather, that the density condition of the Mittag-Leffler procedure implies continuity. Before giving these theorems we present a common generalization of the theorems 3.2.8 and 3.2.9 which is due to Frerick, Kunkle, and the present author [28]. We use the following concept which had been introduced by de Wilde in connection with closed graph theorems [21].

**Definition 3.2.11** *A web in a separated locally convex space $X$ is a system $\mathscr{C} = \{C_{\alpha_1,\dots,\alpha_k} : k, \alpha_1, \dots, \alpha_k \in \mathbb{N}\}$ of absolutely convex subsets of $X$ with*

1. $\bigcup_{n\in\mathbb{N}} C_n = X$,
2. $\bigcup_{n\in\mathbb{N}} C_{\alpha_1,\dots,\alpha_k,n} = C_{\alpha_1,\dots,\alpha_k}$ *for all* $k \in \mathbb{N}$, *and*
3. $\forall\, (\alpha_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}\ \exists\, (\lambda_k)_{k\in\mathbb{N}} \in (0,1)^{\mathbb{N}}\ \forall\, (x_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} C_{\alpha_1,\dots,\alpha_k}$

$$\sum_{k=1}^{\infty} \lambda_k x_k \text{ converges in } X.$$

*For* $\alpha \in \mathbb{N}^{\mathbb{N}}$ *and* $k \in \mathbb{N}$ *we will write* $C_{\alpha|k} = C_{\alpha_1,\ldots,\alpha_k}$. $\mathscr{C}$ *is called ordered if* $\alpha \leq \beta$ *implies* $C_{\alpha|k} \subseteq C_{\beta|k}$ *and strict if the scalars* $\lambda_k$ *in 3. can be chosen in such a way that the limits* $\sum\limits_{k=p}^{\infty} \lambda_k x_k$ *belong to* $C_{\alpha|p}$ *for all* $p$.

We now show that the sequence $(\lambda_k)_{k\in\mathbb{N}}$ can always be taken as $\lambda_k = 2^{-(k+1)}$ independently of $\alpha \in \mathbb{N}^{\mathbb{N}}$. This could be deduced from results of Valdivia [58] but we prefer to give an elementary direct proof.

**Lemma 3.2.12** *Let* $\mathscr{C}$ *be a (strict) web in a separated locally convex space,* $\alpha \in \mathbb{N}^{\mathbb{N}}$ *and* $(x_k)_{k\in\mathbb{N}} \in \prod\limits_{k\in\mathbb{N}} C_{\alpha|k}$. *Then* $\sum\limits_{k=p}^{\infty} \frac{1}{2^{k+1}} x_k$ *converges in* $X$ *(and belongs to* $C_{\alpha|p}$*) for all* $p \in \mathbb{N}$.

**Proof.** Since we will not use property 1. of the web it is enough to show the result for $p = 1$ (for the general case we consider $\widetilde{C}_{\alpha|k} = C_{\alpha|p+k-1}$). Let $(\lambda_k)_{k\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ be such that $\sum_{k=1}^{\infty} \lambda_k y_k$ converges (to an element of $C_{\alpha|1}$) for all $(y_k)_k \in \prod_{k\in\mathbb{N}} C_{\alpha|k}$. We choose a strictly increasing sequence $(l_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $\sum_{j>l_k} 2^{-j} \leq \lambda_{k+1}$. Given $(x_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} C_{\alpha|k}$ we define $y_k = \frac{1}{\lambda_k} \sum\limits_{l_{k-1}<j\leq l_k} 2^{-j} x_j$ which belongs to $C_{\alpha|k}$ since this set is absolutely convex, $x_j \in C_{\alpha|j} \subseteq C_{\alpha|k}$ for $j > l_{k-1} \geq k-1$, and $\sum_{j>l_{k-1}} 2^{-j} \leq \lambda_k$.

Hence $\sum_{k=2}^{\infty} \lambda_k y_k$ converges (to an element of $C_{\alpha|2}$) and thus the subsequence $\sum\limits_{j=1}^{l_n} \frac{1}{2^{j+1}} x_j = \frac{1}{2} \sum\limits_{j=1}^{l_1} \frac{1}{2^j} x_j + \frac{1}{2} \sum\limits_{k=1}^{n} \lambda_k y_k$ of the partial sums converges (to an element in $\frac{1}{2} C_{\alpha|1} + \frac{1}{2} C_{\alpha|2} \subseteq C_{\alpha|1}$). Given $m \in \mathbb{N}$ we choose the maximal $n(m) \in \mathbb{N}$ with $l_{n(m)} \leq m$ and set

$$r_m = \sum_{l_{n(m)}<j\leq m} \frac{1}{2^{j+1}} x_j \in \lambda_{n(m)+1} C_{\alpha|n(m)+1}.$$

We have to show that $r_m$ tends to 0, which would be obvious if the sequence $n(m)$ were strictly increasing. However, if $(m(k))_{k\in\mathbb{N}}$ is any subsequence of the natural numbers there is a further subsequence $(m(k(i))_{i\in\mathbb{N}}$ such that $n(m(k(i)))$ is strictly increasing, hence $r_{m(k(i))} \to 0$ as $i \to \infty$. This shows $r_m \to 0$. □

**Theorem 3.2.13** *Let* $\mathscr{X} = (X_n, \varrho_m^n)$ *be a projective spectrum consisting of separated locally convex spaces and continuous linear maps. If each* $X_n$ *has an ordered web* $\mathscr{C}^n = \{C^n_{\alpha|k} : \alpha \in \mathbb{N}^{\mathbb{N}}, k \in \mathbb{N}\}$ *the following conditions are equivalent.*

1. $\operatorname{Proj}^1 \mathscr{X} = 0$.
2. $\exists\, \alpha \in \mathbb{N}^{\mathbb{N}}\ \forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m$

$$\varrho_m^n X_m \subseteq \varrho_k^n X_k + \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(C^j_{\alpha_j,\ldots,\alpha_n}).$$

*If all webs are strict 1. and 2. are equivalent to*

*3.* $\exists\, \alpha \in \mathbb{N}^{\mathbb{N}}\ \forall\, n \in \mathbb{N}\ \exists\, m \geq n$

$$\varrho^n_m X_m \subseteq \varrho^n(\operatorname{Proj} \mathscr{X}) + \bigcap_{j=1}^{n} (\varrho^j_n)^{-1}(C^j_{\alpha_j,\dots,\alpha_n}).$$

**Proof.** 1. can be derived from 2. as in the first proof of 3.2.1, there the elements $z_n$ can be taken from $\frac{1}{2^{n+1}} \bigcap\limits_{j\leq n} (\varrho^j_n)^{-1}(C^j_{\alpha_j,\dots,\alpha_n})$ and then $r_n = \sum\limits_{k\geq n} \varrho^n_k z_k$ converges in $X_n$ because of 3.2.12. That 2. implies 1. is proved like proposition 3.2.6 (here one uses that the webs are ordered) and 3. follows from 2. as in 3.2.5 since for strict webs $\mathscr{C}^n$ the system

$$\left\{\varepsilon \prod_{j\leq n} C^j_{\alpha_j,\dots,\alpha_n} \times \prod_{j>n} X_j : n \in \mathbb{N},\, \varepsilon > 0\right\}$$

is the basis of a complete metrizable group topology on $\prod\limits_{n\in\mathbb{N}} X_n$. □

If the steps $X_n$ are (LB)-spaces with fundamental sequences $(B_{n,N})_{N\in\mathbb{N}}$ of the Banach discs,

$$\mathscr{C}^n = \big\{\min\{\alpha_1,\dots,\alpha_k\} B_{n,\alpha_1} : k, \alpha_1, \dots \alpha_k \in \mathbb{N}\big\}$$

are a strict ordered webs and 3.2.13 gives 3.2.9.

If the steps $X_n$ are Fréchet spaces with bases $\{U_{n,N} : N \in \mathbb{N}\}$ of $\mathscr{U}_0(X_n)$ the systems

$$\mathscr{C}^n = \left\{\bigcap_{l=1}^{k} \alpha_l U_{n,l} : k, \alpha_1, \dots, \alpha_k \in \mathbb{N}\right\}$$

are strict ordered webs and 3.2.13 reduces to theorem 3.2.8.

Now we present a theorem which was proved by Frerick and the author [29] and (for spectra of (LB)-spaces) independently by Braun and Vogt [19] (both articles dualized arguments from [69]). The present formulation was noticed by M. Langenbruch [42].

$\mathscr{BD}(X)$ denotes the set of Banach discs in a locally convex space $X$.

**Theorem 3.2.14** *Let $\mathscr{X} = (X_n, \varrho^n_m)$ be a projective spectrum consisting of separated locally convex spaces and continuous linear maps such that*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\, B \in \mathscr{BD}(X_n)\ \forall\, M \in \mathscr{BD}(X_m)$$

$$\exists\, K \in \mathscr{BD}(X_k) \quad \varrho^n_m(M) \subseteq \varrho^n_k(K) + B.$$

*Then* $\operatorname{Proj}^1 \mathscr{X} = 0$.

The key point of the proof is the following trick which is behind the open mapping theorem for Banach spaces.

**Lemma 3.2.15** *Let $X$ be a separated topological vector space, $A$ a bounded subset of $X$, and $B \subset X$ a Banach disc. Then $A \subseteq B + \frac{1}{2}A$ implies $A \subseteq 3B$.*

**Proof.** Given $a \in A$ we choose inductively $a_n \in A$ and $b_n \in B$ such that

$$\begin{aligned} a &= b_1 + \frac{1}{2}a_1 = b_1 + \frac{1}{2}(b_2 + \frac{1}{2}a_2) = b_1 + \frac{1}{2}b_2 + \frac{1}{4}a_2 = \ldots = \\ &= \sum_{n=1}^{N} \frac{1}{2^{n-1}} b_n + \frac{1}{2^N} a_N. \end{aligned}$$

The first sum converges in $[B]$ (hence, also in $X$) to $b \in \overline{2B}^{[B]} \subseteq 3B$, and $\frac{1}{2^N}a_N$ tends to 0 in $X$ since $A$ is bounded in $X$. Thus, $a = b \in 3B$. □

**Proof of 3.2.14.** By passing to a subsequence of the spectrum we can assume $m = n + 1$ in the hypothesis. For each $n \in \mathbb{N}$ we choose $B_n \in \mathscr{BD}(X_n)$ according to $k = n + 2$ and set $\widetilde{B}_1 = B_1$. There is $T \in \mathscr{BD}(X_3)$ such that $\varrho_2^1(B_2) \subseteq \varrho_3^1(T) + \widetilde{B}_1$. We set

$$\widetilde{B}_2 = (\varrho_2^1)^{-1}(\widetilde{B}_1) \cap (B_2 + \varrho_3^2(T))$$

which is a Banach disc by 3.2.10 and satisfies $\varrho_2^1(\widetilde{B}_2) \subseteq \widetilde{B}_1$.

Since $B_2 \subseteq \varrho_3^2(T) + (\varrho_2^1)^{-1}(\widetilde{B}_1)$ we find for $x \in B_2$ elements $y \in \varrho_3^2(T)$ and $z \in (\varrho_2^1)^{-1}(\widetilde{B}_1)$ with $x = y + z$. But then $z = x - y \in \widetilde{B}_2$ which shows $B_2 \subseteq \widetilde{B}_2 + \varrho_3^2(T)$. We will now show

$$\forall\, M \in \mathscr{BD}(X_3)\ \exists\, K \in \mathscr{BD}(X_4) \quad \varrho_3^2(M) \subseteq \varrho_4^2(K) + \widetilde{B}_2.$$

Given $M \in \mathscr{BD}(X_3)$ we choose $K \in \mathscr{BD}(X_4)$ such that

$$\begin{aligned} A = 3\varrho_3^2(M + T) &\subseteq \varrho_4^2(K) + B_2 \subseteq \varrho_4^2(K) + \widetilde{B}_2 + \varrho_3^2(T) \\ &\subseteq (\varrho_4^2(K) + \widetilde{B}_2) + \frac{1}{2}A. \end{aligned}$$

The preceding lemma gives $A \subseteq 3(\varrho_4^2(K) + \widetilde{B}_2)$, hence we obtain

$$\varrho_3^2(M) \subseteq \frac{1}{3}A \subseteq \varrho_4^2(K) + \widetilde{B}_2.$$

Modifying $B_n$ for $n \geq 3$ in the same way we construct a sequence $\widetilde{B}_n$ of Banach discs with $\varrho_{n+1}^n(\widetilde{B}_{n+1}) \subseteq \widetilde{B}_n$ and

$$\forall\, M \in \mathscr{BD}(X_{n+1})\ \exists\, K \in \mathscr{BD}(X_{n+2}) \quad \varrho_{n+1}^n(M) \subseteq \varrho_{n+2}^n(K) + \widetilde{B}_n.$$

Using a simple induction and $\varrho_{n+1}^n(\widetilde{B}_{n+1}) \subseteq \widetilde{B}_n$ we obtain

$$\forall\, M \in \mathscr{BD}(X_{n+1}),\ k \geq n+2\ \exists\, K \in \mathscr{BD}(X_k) \quad \varrho_{n+1}^n(M) \subseteq \varrho_k^n(K) + \widetilde{B}_n.$$

Since $X_{n+1}$ is covered by the union of all its Banach discs this implies that the group topologies induced by $\widetilde{B}_n$ satisfy the hypothesis of 3.2.1 which yields $\operatorname{Proj}^1 \mathscr{X} = 0$. □

Theorem 3.2.14 remains true if at all places $\mathscr{BD}(X)$ is substituted by any smaller system $\mathscr{A}(X)$ which is stable under continuous linear images, sums, and scalar multiples, and which covers $X$, e.g. $\mathscr{A}(X)$ may consist of all compact or all weakly compact absolutely convex sets. If $\mathscr{X}$ is the dual of an inductive spectrum (i.e. $X_n = Y_n'$ and $\varrho_m^n = (i_n^m)^t$ for $i_n^m : Y_n \to Y_m$) we may also take the system of weak$^\star$ compact sets.

To interpret the condition of 3.2.14 let us say that the image of $\varrho_k^n$ is dense in $\varrho_m^n(X_m)$ with respect to $A$ (where $A$ is an absolutely convex subset of $X_n$) if $\varrho_m^n(X_m) \subseteq \varrho_k^n(X_k) + A$ (since we can multiply with any $\varepsilon > 0$ this is precisely the density with respect to the group topology on $X_n$ having $\{\varepsilon A : \varepsilon > 0\}$ as a basis of the 0-neighbourhood filter). Similarly, we say that the image of $\varrho_k^n$ is *large* in $\varrho_m^n(X_m)$ with respect to $A$ if

$$\forall\, M \in \mathscr{B}(X_m)\ \exists\, K \in \mathscr{B}(X_k) \quad \varrho_m^n(M) \subseteq \varrho_k^n(K) + A$$

(again we get the same for $\varepsilon A$ instead of $A$, but this condition is not the same as $\varrho_k^n(X_k)$ being a large subspace in the sense of [51, 8.3.22] of $\varrho_m^n(X_m)$ for some topology). With this terminology 3.2.14 says that a spectrum $\mathscr{X}$ of locally complete spaces satisfies $\operatorname{Proj}^1 \mathscr{X} = 0$ if

$\forall\ n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m \quad \varrho_k^n(X_k)$ is large in $\varrho_m^n(X_m)$ with respect to some bounded subset of $X_n$.

It is not known whether this is still true for spectra of (LB)-spaces if “large” is replaced by “dense” (in 3.2.18 we prove this for spectra of (LS)-spaces). The next result from [70] uses a slightly stronger hypothesis, namely that $\varrho^n(\operatorname{Proj} \mathscr{X})$ is dense im $\varrho_m^n(X_m)$ with respect to some bounded set.

**Theorem 3.2.16** *Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a projective spectrum consisting of separated (LB)-spaces and continuous linear maps. Then* $\operatorname{Proj}^1 \mathscr{X} = 0$ *if and only if*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n,\ B_n \in \mathscr{B}(X_n) \quad \varrho_m^n X_m \subseteq \varrho^n(\operatorname{Proj} \mathscr{X}) + B_n.$$

**Proof.** Again, we may assume that $m = n+1$ holds. Let $(B_{n,l})_{l\in\mathbb{N}}$ be fundamental sequences of the Banach discs in $X_n$. We have

$$\varrho_2^1(B_2) \subseteq \varrho^1(\operatorname{Proj} \mathscr{X}) + B_1 = \bigcup_{l\in\mathbb{N}} \varrho^1\left((\varrho^3)^{-1}(B_{3,l})\right) + B_1.$$

Since $Y = [\varrho_2^1(B_2)]$ is a Banach space there is $l_3 \in \mathbb{N}$ such that

$$Y \cap \left\{\varrho^1\left((\varrho^3)^{-1}(B_{3,l_3})\right) + B_1\right\}$$

is not meager in $Y$. But this set equals

$$\bigcup_{l\in\mathbb{N}} Y \cap \left\{\varrho^1\left((\varrho^3)^{-1}(B_{3,l_3}) \cap (\varrho^4)^{-1}(B_{4,l})\right) + B_1\right\},$$

hence there is $l_4 \in \mathbb{N}$ such that

$$Y \cap \left\{\varrho^1\left((\varrho^3)^{-1}(B_{3,l_3}) \cap (\varrho^4)^{-1}(B_{4,l_4})\right) + B_1\right\}$$

is not meager. Inductively we find $(l_j)_{j\geq 3}$ such that

$$Y \cap \left\{\varrho^1\left(\bigcap_{j=3}^{k} (\varrho^j)^{-1}(B_{j,l_j})\right) + B_1\right\}$$

is not meager for every $k \geq 3$ which therefore also holds for the bigger set

$$Y \cap \left\{\varrho^1_k\left(\bigcap_{j=3}^{k} (\varrho^j_k)^{-1}(B_{j,l_j})\right) + B_1\right\}.$$

For $k \geq 3$ the sets $A^1_k := \bigcap\limits_{j=3}^{k} (\varrho^j_k)^{-1}(B_{j,l_j})$ are Banach discs by 3.2.10 and obviously satisfy $\varrho^k_{k+1}(A^1_{k+1}) \subseteq A^1_k$. Since $Y \cap (\varrho^1_k(A^1_k) + B_1)$ is not meager its closure in $Y$ contains an interior point which can be assumed to be 0 by a simple convexity argument. Hence there are $\varepsilon_k \in (0,1)$ with

$$\varepsilon_k \varrho^1_2(B_2) \subseteq \overline{Y \cap (\varrho^1_k(A^1_k) + B_1)} \subseteq \varrho^1_k(A^1_k) + B_1 + \frac{\varepsilon_k}{2}\varrho^1_2(B_2)$$

and lemma 3.2.15 implies

$$\frac{\varepsilon_k}{3}\varrho^1_2(B_2) \subseteq \varrho^1_k(A_k) + B_1 \quad \text{and therefore even}$$

$$\varepsilon^1_k B_2 \subseteq \varrho^2_k(A^1_k) + (\varrho^1_2)^{-1}(B_1) \quad \text{with } \varepsilon^1_k = \frac{\varepsilon_k}{3}.$$

In the same way we find Banach discs $A^n_k$ in $X_k$ and $\varepsilon^n_k \in (0,1)$ for $k \geq n+2$ such that

1. $\varepsilon^n_k B_{n+1} \subseteq \varrho^{n+1}_k(A^n_k) + (\varrho^n_{n+1})^{-1}(B_n)$ and
2. $\varrho^k_{k+1}(A^n_{k+1}) \subseteq A^n_k$ for all $k \geq n+2$.

Setting $A^0_k := \{0\}$ and replacing $A^n_k$ by $A^1_k + \ldots + A^n_k$ if necessary (which would not affect 1. and 2.) we additionally have

3. $A^{n-1}_k \subseteq A^n_k$ for all $k \geq n+2$ and $n \in \mathbb{N}$.

Now we define $\widetilde{B}_1 = B_1$ and inductively

$$\widetilde{B}_{n+1} = \left(\varrho^n_{n+1}\right)^{-1}\left(\widetilde{B}_n\right) \cap \left(B_{n+1} + \varrho^{n+1}_{n+2}\left(A^n_{n+2}\right)\right)$$

which are Banach discs again by 3.2.10.

Proceeding by induction (on $n \in \mathbb{N}$) we show that for all $n \in \mathbb{N}$ and $k \geq n+1$ there are $0 < \delta_k^n \leq 1$ with

$$(*) \qquad \delta_k^n B_n \subseteq \widetilde{B}_n + \varrho_k^n(A_k^{n-1}).$$

This is clear for $n = 1$ with $\delta_k^1 = 1$. Suppose we have found for some $n \in \mathbb{N}$ constants $0 < \delta_k^n \leq 1$ such that $(*)$ holds for all $k \geq n+1$. We set $\delta_k^{n+1} = \frac{1}{2}\varepsilon_k^n \delta_k^n$. Then 1. and the induction hypothesis yield

$$\begin{aligned}\delta_k^{n+1} B_{n+1} &\subseteq \frac{1}{2}\delta_k^n \varrho_k^{n+1}(A_k^n) + \frac{1}{2}(\varrho_{n+1}^n)^{-1}(\delta_k^n B_n) \\ &\subseteq \frac{1}{2}\varrho_k^{n+1}(A_k^n) + \frac{1}{2}(\varrho_{n+1}^n)^{-1}\left(\widetilde{B}_n + \varrho_k^n(A_k^{n-1})\right).\end{aligned}$$

Given $x \in (\varrho_{n+1}^n)^{-1}(\widetilde{B}_n + \varrho_k^n(A_k^{n-1}))$ there are $y \in \widetilde{B}_n$ and $z \in A_k^{n-1}$ such that $\varrho_{n+1}^n(x) = y + \varrho_k^n(z)$, hence $\varrho_{n+1}^n(x - \varrho_k^{n+1}(u)) = y \in \widetilde{B}_n$ which gives $x = x - \varrho_k^{n+1}(z) + \varrho_k^{n+1}(z) \in (\varrho_{n+1}^n)^{-1}(\widetilde{B}_n) + \varrho_k^{n+1}(A_k^{n-1})$. We thus obtain using 3.

$$\begin{aligned}\delta_k^{n+1} B_{n+1} &\subseteq \frac{1}{2}\varrho_k^{n+1}(A_k^n) + \frac{1}{2}\left((\varrho_{n+1}^n)^{-1}(\widetilde{B}_n) + \varrho_k^{n+1}(A_k^{n-1})\right) \\ &\subseteq (\varrho_{n+1}^n)^{-1}(\widetilde{B}_n) + \varrho_k^{n+1}(A_k^n).\end{aligned}$$

Since $\delta_k^{n+1} \leq 1$ this implies for $k \geq n+2$

$$\begin{aligned}\delta_k^{n+1} B_{n+1} &\subseteq \varrho_k^{n+1}(A_k^n) + (\varrho_{n+1}^n)^{-1}(\widetilde{B}_n) \cap \left(B_{n+1} + \varrho_k^{n+1}(A_k^n)\right) \\ &\subseteq \varrho_k^{n+1}(A_k^n) + \widetilde{B}_{n+1} \text{ because of 2.}\end{aligned}$$

To finish the proof we show that the sequence $\widetilde{B}_n$ satisfies condition $\beta$) of theorem 3.2.9. We fix $n \in \mathbb{N}$ and $k > m = n+1$. Multiplying the condition in our assumption by $\delta_k^n$ we obtain

$$\begin{aligned}\varrho_{n+1}^n(X_{n+1}) &\subseteq \varrho^n(\operatorname{Proj} \mathscr{X}) + \delta_k^n B_n \subseteq \varrho_k^n(X_k) + \widetilde{B}_n + \varrho_k^n(A_k^{n-1}) \\ &= \varrho_k^n(X_k) + \widetilde{B}_n.\end{aligned}$$

□

We finish the discussion of the Mittag-Leffler procedure with a result about spectra consisting of (LS)-spaces, i.e. inductive limits of Banach spaces with compact inclusions. We will use the following terminology.

**Definition 3.2.17** *Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a projective spectrum consisting of locally convex spaces and continuous linear maps.*

1. *$\mathscr{X}$ satisfies $(P_3)$ if*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\, B \in \mathscr{B}(X_n)\ \forall\, M \in \mathscr{B}(X_k)$$

$$\exists\, K \in \mathscr{B}(X_k),\ S > 0 \quad \varrho_m^n(M) \subseteq S(\varrho_k^n(K) + B).$$

*2. $\mathscr{X}$ is reduced if*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m \quad \varrho_m^n(X_m) \subseteq \overline{\varrho_k^n(X_k)},$$

*the closure taken in the locally convex space $X_n$.*

$(P_3)$ is a weak variant of a condition $(P_2)$ introduced by Vogt [62] (where the set $B$ should be independent of $k$). For spectra of regular (LB)-spaces $X_n = \operatorname{ind} X_{n,N}$ $(P_3)$ allows a more qualitative formulation:

$$\forall\ n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\ \ N \in \mathbb{N} \quad \varrho_m^n(X_m) \subseteq \varrho_k^n(X_k) + X_{n,N}$$

i.e. $\varrho_k^n(X_k)$ is dense in $\varrho_m^n(X_m)$ with respect to $X_{n,N}$. Necessity of this condition is clear since $X_m$ is covered by the union of its Banach discs and sufficiency follows from Grothendieck's factorization theorem.

If each $X_n$ is a reflexive (LB)-space with fundamental sequence of the bounded sets $(B_{n,N})_{N\in\mathbb{N}}$ we denote by

$$\|y\|_{n,N}^* = \sup\{|\langle y, x\rangle| : x \in B_{n,N}\},\ y \in X_n'$$

the dual seminorms. In this case one can show with the theorem of bipolars that $(P_3)$ is equivalent to the following condition $(P_3^*)$

$$\forall\ n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\, N \in \mathbb{N}\ \forall\, M \in \mathbb{N}\ \exists\, K \in \mathbb{N},\, S > 0$$
$$\forall\, y \in X_n' \quad \|y \circ \varrho_m^n\|_{m,M}^* \leq S(\|y \circ \varrho_k^n\|_{k,K}^* + \|y\|_{n,N}^*).$$

Alternatively, this condition can be formulated with the same quantifiers as follows: For each $y \in X_n'$ (1) and (2) imply (3) with

$$(1)\ \|y\|_{n,N}^* \leq 1 \quad (2)\ \|y \circ \varrho_k^n\|_{k,K}^* \leq 1\ \text{ and } (3)\ \|y \circ \varrho_m^n\|_{m,M}^* \leq S.$$

These inequalities can be helpful to establish surjectivity results for operators on projective limits of (LS)-spaces (the last formulation then gives a kind of Phragmen-Lindelöf condition) because of the following theorem (a version with $(P_2^*)$ had been proved the first time in [69] with a slightly stronger notion of reducedness but for the larger class of (DFM)-spaces). The $(P_2)$-version can also be found in the article of Braun and Vogt [19].

**Theorem 3.2.18** *Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a projective spectrum consisting of (LS)-spaces and continuous linear maps. Then the following conditions are equivalent.*

1. $\operatorname{Proj}^1 \mathscr{X} = 0$.
2. $\forall\ n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\, B \in \mathscr{B}(X_n) \quad \varrho_m^n(X_m) \subseteq \varrho_k^n(X_k) + B.$
3. *$\mathscr{X}$ is reduced and satisfies $(P_3)$.*
4. $\forall\ n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\, B \in \mathscr{B}(X_n)\ \forall\, M \in \mathscr{B}(X_m)$

$$\exists\, K \in \mathscr{B}(X_k) \quad \varrho_m^n(M) \subseteq \varrho_k^n(K) + B.$$

**Proof.** 2. follows from 1. by theorem 3.2.9 and obviously, 2. implies that $\mathscr{X}$ is reduced. $(P_3)$ can be deduced from 2. either by Grothendieck's factorization theorem or using directly a Baire argument as in the proof of 3.2.16.

We now show that for reduced spectra of (LB)-spaces $(P_3)$ implies 2. Let $(B_{n,N})_{N\in\mathbb{N}}$ be fundamental sequences of the bounded sets in $X_n$ and choose for fixed $n\in\mathbb{N}$ an $m\geq n$ as in $(P_3)$ and $\widetilde{m}\geq m$ such that $\varrho^m_{\widetilde{m}}(X_{\widetilde{m}})\subseteq\overline{\varrho^m_k(X_k)}$ for all $k\geq\widetilde{m}$. For $k\geq\widetilde{m}$ there are $B\in\mathscr{B}(X_n)$ and $\varepsilon_M>0$ such that $\varepsilon_M\varrho^n_m(B_{m,M})\subseteq\varrho^n_k(X_k)+B_n$ for all $M\in\mathbb{N}$ which gives $\varrho^n_m(U)\subseteq\varrho^n_k(X_k)+B_n$ where $U=\Gamma(\bigcup\limits_{M\in\mathbb{N}}\varepsilon_M B_{m,M})$ is a neighbourhood of 0 in the (LB)-space $X_m$. Since $\varrho^m_{\widetilde{m}}(X_{\widetilde{m}})\subseteq\varrho^m_k(X_k)+U$ we get

$$\varrho^n_{\widetilde{m}}(X_{\widetilde{m}})\subseteq\varrho^n_m(\varrho^m_k(X_k)+U)\subseteq\varrho^n_k(X_k)+\varrho^n_k(X_k)+B=\varrho^n_k(X_k)+B.$$

We now prove that 3. implies 4. where we may use 2. For $n\in\mathbb{N}$ we choose $m\geq n$ according to 2. and then $\widetilde{m}\geq m$ according to $(P_3)$ for $\widetilde{n}=m$. We fix $k\geq\widetilde{m}$. Then there are $B\in\mathscr{B}(X_n)$ and $D\in\mathscr{B}(X_m)$ such that

$$\varrho^n_k(X_k)\text{ is dense in }\varrho^m_{\widetilde{m}}(X_{\widetilde{m}})\text{ with respect to }[D]$$

$$\text{and }\varrho^n_k(X_k)\text{ is dense in }\varrho^n_m(X_m)\text{ with respect to }B.$$

If $A\in\mathscr{B}(X_m)$ is such that the inclusion $[D]\hookrightarrow[A]$ is compact we claim that $\varrho^n_k(X_k)$ is large in $\varrho^m_{\widetilde{m}}(X_{\widetilde{m}})$ with respect to $\widetilde{B}=B+\varrho^n_m(A)$. Indeed, given $M\in\mathscr{B}(X_{\widetilde{m}})$ there are $K\in\mathscr{B}(X_k)$ and $S>0$ with $\varrho^m_{\widetilde{m}}(M)\subseteq S(\varrho^n_k(K)+D)$. Using the compactness of $[D]\hookrightarrow[A]$ we find a finite set $E\subset X_m$ with $SD\subseteq A+E$ and since $\varrho^n_k(X_k)$ is dense in $\varrho^n_m(X_m)$ with respect to $B$ there is a finite set $\widetilde{E}\subset X_k$ such that $\varrho^n_m(E)\subseteq\varrho^n_k(\widetilde{E})+B$. With $\widetilde{K}=SK+\widetilde{E}$ we obtain $\varrho^m_{\widetilde{m}}(M)\subseteq\varrho^n_k(\widetilde{K})+\widetilde{B}$.

Finally, 4. implies 1. by theorem 3.2.14. □

Let us note that the implications 1. $\Longrightarrow$ 2. $\Longrightarrow$ 3. hold for spectra of separated (LB)-spaces and 3. $\Longrightarrow$ 2. holds for spectra of bornological locally convex spaces. Finally, 3. $\Longrightarrow$ 4. is true for spectra consisting of bornological spaces $X_n$ such that for all $B\in\mathscr{B}(X_n)$ there is $D\in\mathscr{B}(X_n)$ such that the inclusion $[B]\hookrightarrow[D]$ is compact. Strong duals of complete Schwartz spaces are of that type.

Let us finally mention that without reducedness neither $(P_3)$ nor stronger variants like $(P_2)$ imply that $\operatorname{Proj}^1\mathscr{X}$ vanishes as the following example shows which is inspired by an example in [64].

**Example 3.2.19** Let $X_n$ be the space $s'$ of slowly increasing scalar sequences, i.e.

$$X_n=s'=\left\{(\alpha_j)_{j\in\mathbb{N}}\in\mathbb{K}^{\mathbb{N}}:\exists\ N\in\mathbb{N}\quad(\alpha_j j^{-N})_{j\in\mathbb{N}}\text{ is bounded}\right\}$$

which its natural inductive limit topology. These are (LS)-spaces, since the inclusions are even nuclear.

Let $\varrho^n_{n+1} : X_{n+1} \longrightarrow X_n$ be the linear operator which puts a 0 in front of a sequence, i.e. $\varrho^n_{n+1}((a_j)_{j\in\mathbb{N}}) = (0, a_1, a_2, \ldots)$.

With $B = B_n = \{\alpha \in s' : \|\alpha\|_\infty \leq 1\}$ for all $n \in \mathbb{N}$ it is easily seen that the spectrum $\mathscr{X} = (X_n, \varrho^n_m)$ satisfies $\varrho^n_{n+1}(X_{n+1}) \subseteq \varrho^n_k(X_k) + [B]$ which implies $(P_3)$ (in a very strong form since $B_n$ does not even depend on $n$, the quantitative version of this condition was called $(P_1)$ in [64, page 18]). But $\operatorname{Proj}\mathscr{X} = \{0\}$, hence $\operatorname{Proj}^1\mathscr{X} \neq 0$ as follows e.g. from condition $(\gamma)$ in theorem 3.2.9.

This shows that theorem 2.7 of [64] only holds for reduced (DFS)-spectra.

## 3.3 Projective limits of locally convex spaces

In the last section we used topological properties of the steps of a projective spectrum only as a tool for proving results about the algebraic projective limit functor. Now we will consider Proj as a functor acting on projective spectra of locally convex spaces with values in the category of locally convex spaces.

A locally convex projective spectrum is an algebraic spectrum consisting of locally convex spaces and continuous spectral maps. By a morphism we will then mean an algebraic morphism with continuous components. The projective limit $\operatorname{Proj}\mathscr{X} = \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \varrho^n_m(x_m) = x_n,\, n \le m\}$ will always be endowed with the relative topology of the product. $\operatorname{Proj}\mathscr{X}$ is closed in $\prod_{n\in\mathbb{N}} X_n$ if all $X_n$ are Hausdorff, and a basis of $\mathscr{U}_0(\operatorname{Proj}\mathscr{X})$ is given by

$$\left\{(\varrho^n)^{-1}(U) : n \in \mathbb{N},\, U \in \mathscr{U}_0(X_n)\right\}.$$

As a functor on locally convex projective spectra Proj is semi-injective: if

$$0 \longrightarrow \mathscr{X} \stackrel{f}{\longrightarrow} \mathscr{Y}$$

is an exact complex with locally convex spectra $\mathscr{X} = (X_n, \varrho^n_m)$ and $\mathscr{Y} = (Y_n, \sigma^n_m)$ then $f = (f_n)_{n\in\mathbb{N}}$ consists of topological embeddings and this easily implies that $\operatorname{Proj}(f) : \operatorname{Proj}\mathscr{X} \longrightarrow \operatorname{Proj}\mathscr{Y}$ is a topological embedding, too. On the other hand, there are short exact sequences.

$$0 \longrightarrow \mathscr{X} \stackrel{f}{\longrightarrow} \mathscr{Y} \stackrel{g}{\longrightarrow} \mathscr{Z} \longrightarrow 0$$

of locally convex spectra such that $\operatorname{Proj}(g) : \operatorname{Proj}\mathscr{Y} \longrightarrow \operatorname{Proj}\mathscr{Z}$ is not a homomorphism between locally convex spaces (this follows e.g. from the next theorem, an artificial example is in 3.3.2). To measure this "lack of openness" Palamodov introduced the functors $\operatorname{Pr}_M = H_M \circ \operatorname{Proj}$ acting on the category of locally convex spectra with values in the category of linear spaces. This contravariant functor is semi-injective and the additional derived functors $\operatorname{Pr}^+_M(\mathscr{X})$ indicates whether $\operatorname{Proj}\mathscr{Y} \longrightarrow \operatorname{Proj}\mathscr{Z}$ is open onto its range. This follows from theorem 2.2.2, but the next result contains a simple direct proof of the fact that for checking whether $\operatorname{Proj}\mathscr{Y} \longrightarrow \operatorname{Proj}\mathscr{Z}$ is a homomorphism it is enough to do this for the canonical resolution known from section 3.1. This result has been obtained by Palamodov [50, theorem 5.3] via theorem 2.2.2.

**Theorem 3.3.1** *Let $\mathscr{X} = (X_n, \varrho^n_m)$ be a locally convex spectrum. The following conditions are equivalent.*

1. $\Psi_{\mathscr{X}} : \prod_{n\in\mathbb{N}} X_n \longrightarrow \prod_{n\in\mathbb{N}} X_n$ *is open onto its range.*
2. $\forall\, n \in \mathbb{N},\, U \in \mathscr{U}_0(X_n)\ \exists\, m \ge n \quad \varrho^n_m(X_m) \subseteq \varrho^n(\operatorname{Proj}\mathscr{X}) + U.$

*3. For every exact sequence*

$$0 \longrightarrow \mathscr{X} \longrightarrow \mathscr{Y} \longrightarrow \mathscr{Z} \longrightarrow 0$$

*of locally convex spectra the induced map* $\operatorname{Proj}\mathscr{Y} \longrightarrow \operatorname{Proj}\mathscr{Z}$ *is open onto its range.*

**Proof.** If 1. holds and $n \in \mathbb{N}$ and $U \in \mathscr{U}_0(X_n)$ are given we define

$$V = \prod_{j\neq n} X_j \times U \in \mathscr{U}_0(\prod_{n\in\mathbb{N}} X_n).$$

Since $\Psi_{\mathscr{X}}$ is open onto its range there is $W \in \mathscr{U}_0(\prod_{n\in\mathbb{N}} X_n)$ with $W \cap \operatorname{im}\Psi_{\mathscr{X}} \subseteq \Psi_{\mathscr{X}}(V)$. We choose $m \geq n$ such that $W$ contains $\prod_{j<m}\{0\} \times \prod_{j\geq m} X_j$. Given $x_m \in X_m$ we have

$$x = (0,\ldots,x_m,0,\ldots) = \Psi_{\mathscr{X}}(\varrho^1_m(x_m),\ldots,x_m,0,\ldots) \in W \cap \operatorname{im}\Psi_{\mathscr{X}}.$$

Hence there is $v = (v_n)_{n\in\mathbb{N}} \in V$ with $\Psi_{\mathscr{X}}(v) = x$ which gives $\varrho^n_m(x_m) = \varrho^n_m(v_m) - \varrho^n_{m+1}(v_{m+1}) = v_n - \varrho^n_{m+1}(v_{m+1}) \in U + \varrho^n(\operatorname{Proj}\mathscr{X})$. Let now 2. be satisfied and

$$0 \longrightarrow (X_n, \varrho^n_m) \xrightarrow{(f_n)} (Y_n, \sigma^n_m) \xrightarrow{(g_n)} (Z_n, \tau^n_m) \longrightarrow 0$$

be a short exact sequence of locally convex spectra.

Let $U = (\sigma^n)^{-1}(U_n)$ with $U \in \mathscr{U}_0(X_n)$ be a typical neighbourhood of 0 in $\operatorname{Proj}\mathscr{Y}$. We choose $m$ according to $n$ and $f_n^{-1}(U_n)$ and set

$$V = \left(\tau^m\right)^{-1}\left(g_m(V_m)\right) \text{ where } V_m = \frac{1}{2}(\sigma^n_m)^{-1}(U_n).$$

We show $V \cap \operatorname{im} g^* \subseteq g^*(U)$, where $g^*$ is the continuous linear map induced by $(g_n)_{n\in\mathbb{N}}$. Given $z = (z_j)_{j\in\mathbb{N}} \in V \cap \operatorname{im} g^*$ there are $v_j \in Y_j, v_m \in V_m$ and $(y_j)_{j\in\mathbb{N}} \in \operatorname{Proj}\mathscr{Y}$ with $z_j = g_j(v_j) = g_j(y_j)$ for all $j \in \mathbb{N}$. Since $v_m - y_m \in \ker g_m$ there is $a_m \in X_m$ with $v_m - y_m = f_m(a_m)$ and because of 2. there are $x = (x_j)_{j\in\mathbb{N}} \in \operatorname{Proj}\mathscr{X}$ and $u \in \frac{1}{2}f_n^{-1}(U_n)$ with $\varrho^n_m(a_m) = x_n + u$, hence

$$\sigma^n_m(v_m - y_m) = \sigma^n_m(f_m(a_m)) = f_n(\varrho^n_m(a_m)) = f_n(x_n) + f_n(u).$$

Then $z = g^*(y + f^*(x)) = g^*\left((y_j + f_j(x_j))_{j\in\mathbb{N}}\right)$ and for $j \leq n$

$$\begin{aligned} y_n + f_n(x_n) &= \sigma^n_m(y_m) + f_n(x_n) \\ &= \sigma^n_m(v_m) - f_n(u) \in \frac{1}{2}U + \frac{1}{2}U = U. \end{aligned}$$

This shows $z \in g^*(U)$. 1. follows from 3. by considering the particular exact sequence $0 \longrightarrow \mathscr{X} \longrightarrow \mathscr{I} \longrightarrow \mathscr{K} \longrightarrow 0$ from the proof of 3.1.4. □

Let us say $\operatorname{Proj}^+ \mathscr{X} = 0$ if one of the conditions in 3.3.1 holds, which is equivalent to $\operatorname{Pr}^+_M \mathscr{X} = 0$ for all sets $M$ (or just some set which is big enough in the sense of theorem 2.2.2). Comparing 3.3.1 with 3.2.8 we see that for locally convex spectra $\mathscr{X}$ consisting for Fréchet spaces, $\operatorname{Proj}^1 \mathscr{X} = 0$ if and only if $\operatorname{Proj}^+ \mathscr{X} = 0$. Of course, this can be deduced also from the open mapping theorem and the fact that quotients of Fréchet spaces are complete (which itself follows from an appropriate version of the open mapping theorem).

Palamodov asked whether it is always true for a locally convex spectrum $\mathscr{X}$ that $\operatorname{Proj}^1 \mathscr{X} = 0$ implies $\operatorname{Proj}^+ \mathscr{X} = 0$ [50, §12.2]. We will show below that this is indeed true for all spectra whose steps naturally appear in analysis but not in general as the following result shows.

**Example 3.3.2** *There are locally convex projective spectra $\mathscr{X}$ satisfying $\operatorname{Proj}^1 \mathscr{X} = 0$ and $\operatorname{Proj}^+ \mathscr{X} \neq 0$.*

**Proof.** Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces with $Y_{n+1} \subseteq Y_n$ such that the inclusions $\varrho^n_m : Y_m \longrightarrow Y_n$ are continuous and have proper dense range for $m > n$. Then $\mathscr{Y} = (Y_n, \varrho^n_m)$ satisfies $\operatorname{Proj}^1 \mathscr{Y} = 0$ by 3.2.1 and $\operatorname{Proj} \mathscr{Y}$ is isomorphic to $Y = \bigcap_{n\in\mathbb{N}} Y_n$ . Now we set $X_n = Y_n$ but endowed with the finest locally convex topology. Then $\mathscr{X} = (X_n, \varrho^n_m)$ still satisfies $\operatorname{Proj}^1 \mathscr{X} = 0$ since this is an algebraic property, and we will now show that $\operatorname{Proj}^+ \mathscr{X} \neq 0$.

Let $B_0$ be a Hamel basis of $Y$. For each finite set $E \subset X_1$ and all $n \in \mathbb{N}$ we have $X_n \nsubseteq [B_0 \cup E]$ since otherwise we would have $X_m = X_{m+1}$ for some $m \geq n$ which is excluded by our assumptions. Hence we can choose inductively $x_n \in X_n$ which is linearly independent of $B_0 \cup \{x_1, \ldots, x_{n-1}\}$. We extend $B_0 \cup \{x_n : n \in \mathbb{N}\}$ to a basis $B$ of $X_1$ and define a linear functional $f$ on $X_1$ by $f(x_n) = 1$ and $f(b) = 0$ for $b \in B \backslash \{x_n : n \in \mathbb{N}\}$. Since $X_1$ carries the finest locally convex topology $f$ is continuous on $X_1$ and $U = \{x \in X_1 : |f(x)| < 1\}$ is a neighbourhood of 0 in $X_1$. Assuming $\operatorname{Proj}^+ \mathscr{X} = 0$ and using theorem 3.3.1 we find $m \geq 1$ such that $\varrho^1_m(X_m) \subseteq \varrho^1(\operatorname{Proj} \mathscr{X}) + U$, hence $X_m \subseteq Y + U$. In particular, there are $y \in Y$ and $u \in U$ with $x_m = y + u$ which implies the contradiction

$$1 = |f(x_m)| = |f(y) + f(u)| = |f(u)| < 1.$$

□

**Theorem 3.3.3** *Let $\mathscr{X} = (X_n, \varrho^n_m)$ be a locally convex projective spectrum such that each $X_n$ has a strict ordered web. Then $\operatorname{Proj}^1 \mathscr{X} = 0$ implies $\operatorname{Proj}^+ \mathscr{X} = 0$.*

**Proof.** Let $\mathscr{C}^n = \{C^n_{\alpha|k} : \alpha \in \mathbb{N}^{\mathbb{N}}, k \in \mathbb{N}\}$ be strict ordered webs in $X_n$. Theorem 3.2.13 implies that there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ with

$$\varrho^n_m(X_m) \subseteq \varrho^n(\operatorname{Proj} \mathscr{X}) + \bigcap_{j=1}^{n} (\varrho^j_n)^{-1}(C^j_{\alpha_j,\ldots\alpha_n}).$$

We fix $n \in \mathbb{N}$ and $U \in \mathscr{U}_0(X_n)$. Then there are $k \geq n$ and $S > 0$ such that $C^n_{\alpha_n,\ldots,\alpha_k} \subseteq SU$, since otherwise there would be natural numbers $n_l < n_{l+1}$ and $x_l \in \frac{1}{2^{l+1}} C_{\alpha_n,\ldots,\alpha_{n_l}}$ with $x_l \notin U$ for all $l \in \mathbb{N}$ contradicting the convergence of $\sum_{l=1}^{\infty} x_l$ (this is a so-called localization property of webbed spaces). Now, we choose $m$ according to $k$. Then

$$\varrho^n_m(X_m) \subseteq \varrho^n(\operatorname{Proj} \mathscr{X}) + C^n_{\alpha_n,\ldots,\alpha_k} \subseteq \varrho^n(\operatorname{Proj} \mathscr{X}) + SU$$

which gives the conclusion by multiplying with $S^{-1}$. □

Note that according to results of Valdivia [58] every locally complete webbed space has even a strict ordered web. Moreover, the class of spaces having a strict ordered web contains the Banach spaces and is stable with respect to countable inductive and projective limits, sequentially closed subspaces and separated quotients.

Now we investigate linear topological properties of the locally convex space $\operatorname{Proj} \mathscr{X}$. Obviously, it inherits all properties from the steps which are stable with respect to countable products and closed subspaces, in particular, $\operatorname{Proj} \mathscr{X}$ is complete (locally complete, quasi-complete, etc.) or nuclear or Schwartz if so are all $X_n$. The situation is very different for barrelledness properties. We will now present a result of Vogt [62] about projective limits of (LB)-spaces.

**Theorem 3.3.4** *Let $\mathscr{X} = (X_n, \varrho^n_m)$ be a locally convex projective spectrum consisting of separated (LB)-spaces with* $\operatorname{Proj}^1 \mathscr{X} = 0$. *Then* $\operatorname{Proj} \mathscr{X}$ *is ultrabornological.*

**Proof.** According to theorem 3.2.9 there is a sequence of Banach discs $B_n \subset X_n$ such that $\varrho^n_{n+1}(B_{n+1}) \subseteq B_n$ for all $n \in \mathbb{N}$ and

$$(1) \quad \forall\, n \in \mathbb{N}\ \exists\, m \geq n \quad \varrho^n_m(X_m) \subseteq \varrho^n(\operatorname{Proj} \mathscr{X}) + B_n.$$

The proof is divided into several steps. We first show

$$(2) \quad \forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, D \in \mathscr{BD}(X_m)\ \exists\, A\mathscr{BD}(X_n)$$

$$\varrho^n_m(D) \subseteq \overline{A \cap \varrho^n(\operatorname{Proj} \mathscr{X})}^{[A]}.$$

For this end we choose $m \geq n$ as in (1) and set $A = \varrho^n_m(D) + B_n$ . For each $0 < \varepsilon < 1$ we then have

$$\varrho^n_m(D) \subseteq \Big(\varrho^n(\operatorname{Proj} \mathscr{X}) + \varepsilon B_n\Big) \cap \varrho^n_m(D) \subseteq \Big(\varrho^n\Big(\operatorname{Proj} \mathscr{X}\Big) \cap A\Big) + \varepsilon A.$$

which implies (2). In what follows we will write $A^\infty := (\varrho^n)^{-1}(A)$ for any subset $A \subseteq X_n$ (it will always be clear from the context to which step $A$ belongs). Now, we claim

$$(3) \quad \forall\, (A_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{BD}(X_n)\ \exists\, (D_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{BD}(X_n),\ (S_n)_n \in (0,\infty)^{\mathbb{N}}$$

$$\forall\, n \le N \quad A_n^\infty \subseteq S_N\Bigg(\bigcap_{j=1}^{n} D_j^\infty + \bigcap_{j=n+1}^{N} D_j^\infty\Bigg).$$

The set $A = \prod_{n\in\mathbb{N}} A_n$ is a Banach disc in $\prod_{n\in\mathbb{N}} X_n$ and since $\Psi_{\mathscr{X}}$ is surjective we can inductively find $D_n \in \mathscr{BD}(X_n)$ such that

$$[A] \cap \Psi_{\mathscr{X}}\Bigg(\prod_{n\le N} D_n \times \prod_{n>N} X_n\Bigg)$$

is not meager, hence the closures of these sets in the Banach space [A] contain 0 as an interior point. Using lemma 3.2.15 we find $\varepsilon_N > 0$ such that

$$\varepsilon_N A \subseteq \Psi\Bigg(\prod_{n\le N} D_n \times \prod_{n>N} X_n\Bigg).$$

As in lemma 3.2.4 this gives $\varepsilon_N A_n \subseteq \bigcap\limits_{j=1}^{n} D + \bigcap\limits_{j=n+1}^{N} \varrho^n(D_j^\infty)$ which easily implies (3).

Let now $M \subseteq \operatorname{Proj} \mathscr{X}$ be an absolutely convex set which absorbs all Banach discs. Then we have

$$(4) \quad \exists\, n \in \mathbb{N}\ \forall\, A \in \mathscr{BD}(X_n)\ \exists\, S > 0 \quad A^\infty \subseteq SM.$$

Assuming the contrary, we find $A_n \in \mathscr{BD}(X_n)$ such that $A_n^\infty$ is not absorbed by $M$. We choose $D_n$ and $S_n$ according to (3) and claim that there is $n \in \mathbb{N}$ such that $M$ absorbs $\bigcap\limits_{j=1}^{n} D_j^\infty$. Otherwise there would be $x_n \in \bigcap\limits_{j=1}^{n} D_j^\infty$ with $x_n \notin nM$. For $n \ge m$ we have $\varrho^m(x_n) \in D_m$, hence

$$\varrho^m(x_n) \in \widetilde{D}_m := D_m + \Gamma\left(\{\varrho^m x_j : j < m\}\right) \in \mathscr{BD}(X_m) \text{ for all } n \in \mathbb{N}.$$

This gives $\{x_n : n \in \mathbb{N}\} \subseteq \prod\limits_{m\in\mathbb{N}} \widetilde{D}_m \cap \operatorname{Proj} \mathscr{X}$ which is a Banach disc and thus it is absorbed by $M$, a contradiction. The same argument gives some $N \ge n$ such that $\bigcap\limits_{j=n}^{N} D^\infty$ is absorbed by $M$, and since $M$ is absolutely convex it absorbs $A_n^\infty \subseteq S_N(\bigcap\limits_{j=1}^{n} D_j^\infty + \bigcap\limits_{j=n+1}^{N} D_j^\infty)$. This contradiction shows (4).

Let now $(E, \|\cdot\|)$ be a Banach space and $T : \operatorname{Proj} \mathscr{X} \longrightarrow E$ a linear operator which is bounded on the Banach discs, hence (4) applies to $M = \{x \in \operatorname{Proj} \mathscr{X} : \|Tx\| \le 1\}$. We choose $n$ according to (4) and then $m \ge n$ according to (2). We have

$$\ker \varrho^n = \bigcap \Big\{\varepsilon A^\infty : \varepsilon > 0,\ A \in \mathscr{BD}(X_n)\Big\} \subseteq \bigcap_{\varepsilon>0} \varepsilon M = \ker T,$$

hence there is a unique linear operator $\widetilde{T} : \varrho^n(\operatorname{Proj} \mathscr{X}) \longrightarrow E$ with $\widetilde{T} \circ \varrho^n = T$. For $A \in \mathscr{B}\mathscr{D}(X_n)$ the set $\widetilde{T}(A \cap \varrho^n(\operatorname{Proj} \mathscr{X}))$ is bounded in $E$ since $M$ absorbs $A^\infty$. We can extend $\widetilde{T}$ uniquely to a continuous linear operator on

$$\left[\overline{\varrho^n(\operatorname{Proj} \mathscr{X}) \cap A}^{[A]}\right] =: X_A.$$

For $A \subseteq A_1$ these extensions coincide on $X_A$, hence if $Z$ is the inductive limit of the family $\{X_A : A \in \mathscr{B}\mathscr{D}\}$ we defined a continuous linear operator

$$\hat{T} : Z \longrightarrow E \text{ with } \hat{T} \circ \varrho^n = T.$$

Now, (2) means that $\varrho^n_m : X_m \longrightarrow X_n$ takes values in $Z$ and is continuous as an operator $X_m \longrightarrow Z$ since $X_m$ is ultrabornological. This finally implies that $T = \hat{T} \circ \varrho^n_m \circ \varrho^m$ is continuous. □

For spectra of (LS)-spaces there is a very different proof of 3.3.4 based on the duality to be explained in chapter 6. Because of theorems 3.2.18 and 6.4 the strong dual $(\operatorname{Proj} \mathscr{X})'_\beta$ is an acyclic (LF)-space with Fréchet-Schwartz steps which is complete by corollary 6.5 and a Schwartz space (since this class is stable with respect to inductive limits). This implies that $\operatorname{Proj} \mathscr{X} = (\operatorname{Proj} \mathscr{X})''$ is ultrabornological.

Let us remark that in the case where all $X_n$ are even Banach spaces $\operatorname{Proj} \mathscr{X}$ is a Fréchet space, hence ultrabornological independently of whether $\operatorname{Proj}^1 \mathscr{X} = 0$ or not. To show at least a partial converse of 3.2.4 we will need another notion.

**Definition 3.3.5** *A locally convex projective spectrum $\mathscr{X} = (X_n, \varrho^n_m)$ is strongly reduced if for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that*

$$\varrho^n_m(X_m) \subseteq \overline{\varrho^n(\operatorname{Proj} \mathscr{X})}.$$

Before comparing this with the reducedness property defined in 3.2.17 we give the following result of [62]. As on page 35 we use the dual norms $\|y\|^*_{n,N} = \sup_{x \in B_{n,N}} |y(x)|$ for $y \in X'_n$ where $(B_{n,N})_{n \in \mathbb{N}}$ is a fundamental sequence of bounded sets in the (LB)-space $X_n$.

**Theorem 3.3.6** *If $\mathscr{X} = (X_n, \varrho^n_m)$ is a strongly reduced locally convex spectrum of separated (LB)-spaces such that* $\operatorname{Proj} \mathscr{X}$ *is barrelled, then $\mathscr{X}$ satisfies $(P_2^*)$:*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n,\ N \in \mathbb{N}\ \forall\, k \geq m,\ M \in \mathbb{N}\ \exists\, K \in \mathbb{N},\ S > 0$$

$$\forall\, y \in X'_n \quad \|y \circ \varrho^n_m\|^*_{m,M} \leq S\left(\|y \circ \varrho^n_k\|^*_{k,K} + \|y\|^*_{n,N}\right)$$

$(P_2^*)$ is clearly stronger than $(P_3^*)$ defined after 3.2.17 where $N$ may depend on $k$. Since we deal with dual conditions, it is not surprising that the proof of 3.3.6 uses a bit duality theory. Let us first explain this. For any locally convex spectrum $\mathscr{X} = (X_n, \varrho_m^n)$ the dual $Y = (\text{Proj}\, \mathscr{X})'$ is the union $\bigcup_{n\in\mathbb{N}} (\varrho^n)^t(X_n')$. If the steps are (LB-spaces), $X_n'$ endowed with the strong topologies are Fréchet spaces, hence $Y$ endowed with the finest locally convex topology such that all $(\varrho^n)^t : X_n' \longrightarrow Y$ are continuous is an (LF)-space. However, good results for (LF)-spaces require injective inductive spectra. We will give quite a few results about (LF)-spaces in chapter 6 but let us already state one proposition, due to Vogt [65], which is needed for 3.3.6. The proof is taken from [69].

**Proposition 3.3.7** *Let $Y_n$ be Fréchet spaces with fundamental sequences $(\|\ \|_{n,N})_{N\in\mathbb{N}}$ of seminorms and let $i_n^m : Y_n \longrightarrow Y_m$ be injective linear continuous maps with $i_m^k \circ i_n^m = i_n^k$ and $i_n^n = id_{Y_n}$ such that the inductive limit $Y = \text{ind}Y_n$ is separated and locally complete. Then*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n,\ N \in \mathbb{N}\ \forall\, k \geq m,\ M \in \mathbb{N}\ \exists\, K \in \mathbb{N},\ S > 0\ \forall\ y \in Y_n$$

$$\|i_n^m y\|_{m,M} \leq S(\|i_n^k y\|_{k,K} + \|y\|_{n,N})$$

**Proof.** By Grothendieck's factorization theorem, every bounded set in $Y$ is contained and bounded in some $Y_n$. We fix $n \in \mathbb{N}$ and set

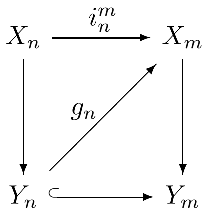
$$U_{n,M} = \{x \in Y_n : \|x\|_{n,M} \leq 1\}\,.$$

We claim that there is $m \geq n$ such that $i_n^m(B)$ is bounded in $Y_m$ whenever $B$ is contained in $U_{n,m}$ with $i_n(B)$ bounded in $Y$ (where $i_n : Y_n \longrightarrow Y$ is the canonical inclusion). If this were false we could find $B_m \subseteq U_{n,m}$ with $i_n(B_m)$ bounded in $Y$ and $i_n^m(B_m)$ unbounded in $Y_m$. But $B = \bigcup_{m\in\mathbb{N}} B_m$ is bounded in $Y$ since for each $U \in \mathscr{U}_0(Y)$ there is $m$ with $i_n(U_{n,m}) \subseteq U$. Hence, there is $m \in \mathbb{N}$ such that $B$ is bounded in $Y_m$ in contradiction to the unboundedness of $B_m \subseteq B$.

Now, we take $N = m$, fix $k \geq m$ and $M \in \mathbb{N}$, and assume that for all $K \in \mathbb{N}$ there are $y_K \in Y_n$ with $\|i_n^m y_K\|_{m,M} > K(\|i_n^k y_K\|_{k,K} + \|y_K\|_{n,N})$.

We put $M_K = \|i_n^k y_K\|_{k,K} + \|y_K\|_{n,N}$. If all but finitely many $M_K$ were 0, $L = [\{i_n(y_K) : K \geq K_0\}]$ would be a bounded subspace of $Y$ which yields $L = \{0\}$ and $y_K = 0$ since $i_n$ is injective. We may thus assume without loss of generality $M_K \neq 0$ for all $K \in \mathbb{N}$ and then $\widetilde{y}_K = M_K^{-1} y_K$ is a sequence in $U_{n,M}$ with $i_n(\widetilde{y}_K)$ bounded in $Y$, and this implies $\|i_n^m \widetilde{y}_K\|_{m,M} \leq S$ for some $S > 0$ and all $K \in \mathbb{N}$. □

**Proof of 3.3.6.** Let $Y = (\text{Proj}\, \mathscr{X})'$ be endowed with the (LF)-space topology described above. Then the identity map $Y \longrightarrow (\text{Proj}\, \mathscr{X})'_\beta$ is continuous which implies that $Y$ is separated. As a strong dual of a barrelled space, $(\text{Proj}\, \mathscr{X})'_\beta$ is locally complete and this gives that $Y$ is locally complete. If the transposed maps $(\varrho^n)^t : X_n' \longrightarrow Y$ were injective, 3.3.6 would follow immediately from 3.3.7. However, since $\mathscr{X}$ is strongly reduced the injective inductive spectrum formed by the spaces Fréchet spaces $X_n'/\ker(\varrho^n)^t =: Y_n$ is equivalent to the spectrum formed by the spaces $X_n'$ in the sense that there are $m = m(n) > n$ and $g_n : Y_n \longrightarrow X_m'$ such that the diagrams



are commutative. Moreover, it is easily seen that the condition in 3.3.7 is invariant under passing to equivalent spectra. This gives the conclusion as $(P_2^*)$ is just the condition of 3.3.7 for the spectrum $(X_n', (\varrho_m^n)^t)$. $\square$

Let us make some remarks on strongly reduced locally convex spectra. Of course, such spectra are reduced and theorem 3.2.9 implies that locally convex spectra $\mathscr{X}$ of separated (LB)-spaces are strongly reduced if $\operatorname{Proj}^1 \mathscr{X} = 0$ holds. If $\mathscr{X}$ consists of Banach spaces then $\mathscr{X}$ is strongly reduced iff $\mathscr{X}$ is reduced iff $\operatorname{Proj}^1 \mathscr{X} = 0$ holds. Moreover, a locally convex spectrum $\mathscr{X} = (X_n, \varrho_m^n)$ is strongly reduced if and only if

$$\forall\, n \in \mathbb{N}\ \exists\, m \in \mathbb{N}\ \forall\, U \in \mathscr{U}_0(X_n) \quad \varrho_m^n(X_m) \subseteq \varrho^n(\operatorname{Proj} \mathscr{X}) + U.$$

Theorem 3.3.1 immediately implies that $\operatorname{Proj}^+ \mathscr{X} = 0$ holds for strongly reduced spectra. Another important property of strongly reduced spectra is contained in the following result mentioned in [64].

**Proposition 3.3.8** *Let $\mathscr{X} = (X_n, \varrho_m^n)$ and $\mathscr{Y} = (Y_n, \sigma_m^n)$ be two strongly reduced spectra of complete separated (LB)-spaces, and $T : \operatorname{Proj} \mathscr{X} \longrightarrow \operatorname{Proj} \mathscr{Y}$ a continuous linear map. Then there is a morphism of locally convex spectra $\widetilde{T} : \widetilde{\mathscr{X}} \longrightarrow \mathscr{Y}$, where $\widetilde{\mathscr{X}}$ is a subsequence of $\mathscr{X}$, such that $T = \operatorname{Proj}(\widetilde{T})$. In particular, $\operatorname{Proj} \mathscr{X} \cong \operatorname{Proj} \mathscr{Y}$ implies that $\mathscr{X}$ and $\mathscr{Y}$ are equivalent.*

**Proof.** We first show

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, U \in \mathscr{U}_0(Y_n)\ \exists\, V \in \mathscr{U}_0(X_m)$$
$$T\left( (\varrho^m)^{-1}(V) \right) \subseteq (\sigma^n)^{-1}(U).$$

Assuming the contrary we find $n \in \mathbb{N}$ and a sequence $U_m \in \mathscr{U}_0(Y_n)$ such that $T\left( (\varrho^m)^{-1}(V) \right)$ is not contained in $(\sigma^n)^{-1}(U_m)$ for all $m \in \mathbb{N}$ and all $V \in \mathscr{U}_0(X_m)$. Since (LB)-spaces have the countable neighbourhood property there are $S_m > 0$ such that $U = \bigcap_{m \in \mathbb{N}} S_m U_m$ is a neighbourhood of 0 in $Y_n$. The continuity of $\sigma^n \circ T$ implies that there are $m_0 \in \mathbb{N}$ and $V \in \mathscr{U}_0(X_{m_0})$ such that $\sigma^n \circ T\left( (\varrho^{m_0})^{-1}(S_{m_0} V) \right)$ is contained in $U \subseteq S_{m_0} U_{m_0}$ which contradicts the choice of $U_{m_0}$.

Since $Y$ is separated, $T$ vanishes on $\ker \varrho^m$ and induces a unique continuous linear map $T' : \operatorname{im}(\varrho^m) \longrightarrow Y_n$ with $T \circ \varrho^m = T'$, and since $Y$ is complete we can extend $T'$ to $\overline{\operatorname{im}(\varrho^m)}^{X_m} \supseteq \varrho_k^m X_k$ for some $k \geq m$ and thus obtain $\widetilde{T}_k = T' \circ \varrho_k^m : X_k \longrightarrow Y_n$ with $\widetilde{T}_k \circ \varrho^k = \sigma^n \circ T$. $\square$

We now show that theorem 3.3.6 is not true for locally convex spectra $\mathscr{X}$ (consisting even of nuclear (LB)-spaces) which are merely reduced.

**Example 3.3.9** There are injective inductive spectra $(Z_n, i_n^m)$ of nuclear Fréchet spaces with non-separated inductive limit $Z$, see e.g. [27, §5, corollary 2]. Since $Y = \overline{\{0\}}^Z$ is complemented in $Z$ we have $Y = \operatorname{ind} Y_n$ with $Y_n = Z_n \cap Y$. Hence there are injective inductive spectra $(Y_n, j_n^m)$ of nuclear Fréchet spaces such that the inductive limit carries the coarsest topology. Let now $X_n = Y_n'$ be endowed with the strong topology and $\varrho_m^n = (j_n^m)^t$. Then $\mathscr{X} = (X_n, \varrho_m^n)$ is a locally convex projective spectrum with $\operatorname{Proj} \mathscr{X} \cong Y' = \{0\}$. Moreover, $\varrho_m^n$ has dense range since $(\varrho_m^n)^t = (j_n^m)^{tt} = j_n^m$ is injective. Therefore, $\mathscr{X}$ is reduced with barrelled projective limits but does not satisfy $(P_2^*)$ because of theorem 3.2.18.

Let us state explicitely the combination of 3.3.4, 3.3.6 and 3.2.18 for spectra of (LS)-spaces.

**Corollary 3.3.10** *Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a locally convex projective spectrum of (LS)-spaces. Then the following conditions are equivalent.*

1. $\operatorname{Proj}^1 \mathscr{X} = 0$.
2. *$\mathscr{X}$ is reduced and satisfies $(P_3^*)$.*
3. *$\mathscr{X}$ is strongly reduced with ultrabornological projective limit.*
4. *$\mathscr{X}$ is strongly reduced with barrelled projective limit.*

Of course, we can replace barrelledness in 4. by any weaker condition which implies that $(\operatorname{Proj} \mathscr{X})'$ endowed with its (LF)-space topology is locally complete, e.g. $\aleph_0$-barrelledness.

Now we are going to study when a projective limit inherits quasinormability from the steps. It is known that a Fréchet space $X$ which is the limit of a reduced projective spectrum $\mathscr{X}$ consisting of Banach spaces is quasinormable if and only if $\Psi_{\mathscr{X}}$ lifts bounded sets (a simple proof of this fact is contained in chapter 7). Let us investigate this property first.

We define for a locally convex space $X$ and a set $I$ the space $\ell_I^\infty(X) = \{(x_i)_{i \in I} \in X^I : \{x_i : i \in I\} \text{ is bounded in } X\}$ endowed with the locally convex topology having $\{U^I \cap \ell_I^\infty(X) : U \in \mathscr{U}_0(X)\}$ as a basis of the 0-neighbourhood filter. If $f : X \longrightarrow Y$ is a continuous linear map we define $\ell_I^\infty(f) : \ell_I^\infty(X) \longrightarrow \ell_I^\infty(Y)$ by $(x_i)_{i \in I} \longrightarrow (f(x_i))_{i \in I}$. In chapter 7 we will consider $\ell_I^\infty$ as a functor acting on the category of locally convex spaces, but for the moment we need it only as a tool. Note that each bounded subset of $\ell_I^\infty(X)$ is contained in $B^I$ for some $B \in \mathscr{B}(X)$.

If $\mathscr{X} = (X_n, \varrho_m^n)$ is a locally convex projective spectrum we denote by $\ell_I^\infty(\mathscr{X})$ the spectrum $(\ell_I^\infty(X_n), \ell_I^\infty(\varrho_m^n))$. With this notion we have

**Theorem 3.3.11** *For a locally convex projective spectrum $\mathscr{X} = (X_n, \varrho_m^n)$ the following conditions are equivalent.*

1. *$\Psi_{\mathscr{X}}$ lifts bounded sets.*
2. *For each exact sequence*

$$0 \longrightarrow \mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z} \longrightarrow 0$$

*of locally convex spectra where the components of $g$ lift bounded sets the induced map* $\operatorname{Proj}\mathscr{Y} \longrightarrow \operatorname{Proj}\mathscr{Z}$ *lifts bounded sets.*
3. $\operatorname{Proj}^1 \ell_I^\infty(\mathscr{X}) = 0$ *for each set $I$.*

**Proof.** 3. follows from 1. by the fact that a continuous linear map $f$ lifts bounded sets iff $\ell_I^\infty(f)$ is surjective for each set $I$ and using the canonical isomorphism $\prod\limits_{n\in\mathbb{N}} \ell_I^\infty(X_n) \cong \ell_I^\infty(\prod\limits_{n\in\mathbb{N}} X_n)$. 2. follows from 3. since

$$0 \longrightarrow \ell_I^\infty(\mathscr{X}) \xrightarrow{\widetilde{f}} \ell_I^\infty(\mathscr{Y}) \xrightarrow{\widetilde{g}} \ell_I^\infty(\mathscr{Z}) \longrightarrow 0$$

(with $\widetilde{f} = \ell_I^\infty(f)$ and $\widetilde{g} = \ell_I^\infty(g)$) is exact as a sequence of projective spectra of vector spaces, and 2. implies 1. by considering the canonical resolution of $\mathscr{X}$ as in the proof of 3.1.4. □

**Proposition 3.3.12** *Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a locally convex projective spectrum. If $\Psi_{\mathscr{X}}$ lifts bounded sets, then*

$$\forall (B_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{B}(X_n)\ \exists\, (D_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{B}(X_n)\ \forall n \in \mathbb{N}$$

$$B_n \subseteq \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(D_j) + \varrho^n\Big(\bigcap_{j>n} (\varrho^j)^{-1}(D_j)\Big).$$

*If all spaces are locally complete the converse implication holds, too.*

**Proof.** For $(B_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{B}(X_n)$ there is $(D_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{B}(X_n)$ with $\prod_{n\in\mathbb{N}} B_n \subseteq \Psi_{\mathscr{X}}(\prod_{n\in\mathbb{N}} D_n)$. If $n \in \mathbb{N}$ and $b_n \in B_n$ are given we set $b_j = 0$ for $j \neq n$. Then there is $d = (d_n)_{n\in\mathbb{N}} \in \prod\limits_{n\in\mathbb{N}} D_N$ with $b = (b_j)_{j\in\mathbb{N}} = \Psi_{\mathscr{X}}(d)$. This gives

$$b_n = d_n - \varrho_{n+1}^n d_{n+1} \in \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(D_j) + \varrho^n\left(\bigcap_{j>n} (\varrho^j)^{-1}(D_j)\right)$$

since $d_j = \varrho_{j+1}^j(d_{j+1})$ for $j \neq n$. To prove the converse implication we choose for a sequence $(B_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{B}(X_n)$ a sequence $(D_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{B}(X_n)$ of closed absolutely convex sets such that

$$2^n B_n \subseteq \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(D_j) + \varrho^n\Big(\bigcap_{j>n} (\varrho^i)^{-1}(D_j)\Big)$$

for all $n \in \mathbb{N}$, and we set $D = \prod_{n\in\mathbb{N}} D_n$ which is a Banach disc in $\prod_{n\in\mathbb{N}} X_n$ since this space is locally complete. Given $b = (b_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} B_n$ there are $x_n \in \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(D_j)$ and $y_n \in \bigcap_{j>n} (\varrho^j)^{-1}(D_j)$ with $2^n b_n = x_n - \varrho^n(y_n)$, hence $(0,\ldots,b_n,0,\ldots) = \frac{1}{2^n}\Psi(d^n)$ where

$$d^n = (\varrho^1(x_n),\ldots,x_n,\varrho^{n+1}(y),\varrho^{n+2}(y),\ldots) \in D.$$

The series $d = \sum_{n=1}^{\infty} \frac{1}{2^n} d^n$ converges in $[D]$ to an element $d \in D$ and hence, it also converges in $\prod_{n\in\mathbb{N}} X_n$. Since $\Psi_{\mathscr{X}}$ is continuous we obtain

$$\Psi_{\mathscr{X}}(d) = \sum_{n=1}^{\infty}(0,\ldots,b_n,0,\ldots) = b.$$

□

**Theorem 3.3.13** *Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a locally convex projective spectrum such that $\Psi_{\mathscr{X}}$ lifts bounded sets. If all $X_n$ are quasinormable then so is* $\operatorname{Proj}\mathscr{X}$.

**Proof.** We first claim

(1) $\forall\, n \in \mathbb{N},\, U_n \in \mathscr{U}_0(X_n)\ \exists\, m \geq n\ \forall\, B \in \mathscr{B}(X_m)\ \exists\, D \in \mathscr{B}(\operatorname{Proj}\mathscr{X})$

$$B \subseteq \varrho^m(D) + (\varrho_m^n)^{-1}(U_n).$$

Otherwise there would be $n \in \mathbb{N}$, $U_n \in \mathscr{U}_0(X_n)$, and $(B_m)_{m\geq n} \in \prod_{m\geq n} \mathscr{B}(X_m)$ such that $B_m \not\subseteq \varrho^m(D) + (\varrho_m^n)^{-1}(U_n)$ for all $D \in \mathscr{B}(\operatorname{Proj}\mathscr{X})$ and all $m \geq n$. We set $B_j = \{0\}$ for $j < n$ and apply 3.3.12 to find $(D_m)_{m\in\mathbb{N}} \in \prod_{m\in\mathbb{N}} \mathscr{B}(X_m)$ with

$$mB_m \subseteq \bigcap_{j\leq m} (\varrho_m^j)^{-1}(D_j) + \varrho^m(K_m)$$

where $K_m = \bigcap_{j>m} (\varrho^j)^{-1}(D_j)$ is bounded in $\operatorname{Proj}\mathscr{X}$ for each $m \in \mathbb{N}$. We choose $m \in \mathbb{N}$ such that $\frac{1}{m} D_n \subseteq U_n$ and set $D = K_m$. Then

$$\varrho_m^n(B_m) \subseteq \frac{1}{m} D_n + \varrho^n(D) \subseteq U_n + \varrho^n(D).$$

For $x \in B_m$ there are $u \in U_n, d \in D$ with $\varrho_m^n(x) = u + \varrho^n(d)$, hence

$$x = x - \varrho^m(d) + \varrho^m(d) \in (\varrho_m^n)^{-1}(U_n) + \varrho^m(D).$$

This proves (1). Let now $U \in \mathscr{U}_0(\operatorname{Proj} X)$ be given. We have to find $V \in \mathscr{U}_0(\operatorname{Proj}\mathscr{X})$ such that for each $\varepsilon > 0$ there is $D \in \mathscr{B}(X)$ with $V \subseteq \varepsilon U + D$. There are $n \in \mathbb{N}$ and $U_n \in \mathscr{U}_0(X_n)$ with $(\varrho^n)^{-1}(U_n) \subseteq U$. We choose $m \geq n$ according to (1) and $V_m \in \mathscr{U}_0(X_m)$ such that for each $\varepsilon > 0$ there is $B \in$

$\mathscr{B}(X_m)$ with $V_m \subseteq \varepsilon(\varrho_m^n)^{-1}(U_n)+B$ which is possible by the quasinormability of $X_m$. Let now $V = (\varrho^m)^{-1}(V_m) \in \mathscr{U}_0(\operatorname{Proj}\mathscr{X})$ and $\varepsilon > 0$. We choose $B \in \mathscr{B}(X_m)$ with $V_m \subseteq \frac{\varepsilon}{2}(\varrho_m^n)^{-1}(U_n) + B$ and then $D \in \mathscr{B}(\operatorname{Proj}\mathscr{X})$ with $\frac{2}{\varepsilon}B \subseteq (\varrho_m^n)^{-1}(U_n) + \varrho^m(D)$. Hence $V_m \subseteq \varepsilon(\varrho_m^n)^{-1}(U_n) + \frac{\varepsilon}{2}\varrho^m(D)$ and this implies $V \subseteq \varepsilon U + \frac{\varepsilon}{2}D$. □

Let us give two more results about the lifting of bounded sets.

**Theorem 3.3.14** *Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a locally convex projective spectrum of locally complete spaces. Then the condition*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\, B \in \mathscr{B}(X_n)\ \forall\, M \in \mathscr{B}(X_m)\ \exists\, K \in \mathscr{B}(X_k)$$

$$\varrho_m^n(M) \subseteq \varrho_k^n(K) + B$$

*implies that $\Psi_{\mathscr{X}}$ lifts bounded sets and there are $\widetilde{B}_n \in \mathscr{B}(X_n)$ with $\varrho_m^n(\widetilde{B}_m) \subseteq \widetilde{B}_n$ for $m \geq n$ and*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, M \in \mathscr{B}(X_m)\ \exists\, D \in \mathscr{B}(\operatorname{Proj}\mathscr{X})$$

$$\varrho_m^n(M) \subseteq \varrho^n(D) + \widetilde{B}_n.$$

**Proof.** Using the fact that all bounded sets of $\ell_I^\infty(X)$ are contained in sets of the form $B^I$ with $B \in \mathscr{B}(X)$ we easily obtain that the spectrum $\ell_I^\infty(\mathscr{X})$ satisfies the same condition as $\mathscr{X}$. Since $\ell_I^\infty(X)$ is locally complete if so is $X$ the result follows from 3.2.14 and its proof. □

For spectra of regular (LB)-spaces we even have:

**Corollary 3.3.15** *Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a locally convex projective spectrum of regular (LB)-spaces. Then $\Psi_{\mathscr{X}}$ lifts bounded sets if and only if the condition of 3.3.14 holds.*

**Proof.** If we endow $\ell_I^\infty(X_n)$ with the inductive limit topology induced by the spectrum $\{\ell_I^\infty(X_{n,N}),\ N \in \mathbb{N}\}$ we obtain regular (LB)-spaces to which we can apply theorem 3.2.9 which easily gives the necessity of the condition in 3.3.14. □

**Corollary 3.3.16** *If $\mathscr{X}$ is a locally convex projective spectrum of (LS)-spaces then* $\operatorname{Proj}^1\mathscr{X} = 0$ *if and only if $\Psi_{\mathscr{X}}$ lifts bounded sets.*

Let us finally mention that there is no hope to extend the converse of theorem 3.3.13 beyond the class of reduced spectra consisting of Banach spaces. If $\mathscr{X} = (X_n, \varrho_m^n)$ is a locally convex spectrum consisting of Schwartz spaces then $\operatorname{Proj}\mathscr{X}$ is again a Schwartz space (since this class is stable with respect to cartesian products and subspaces ) and hence quasinormable.

## 3.4 Some Applications

In this section we give some examples of surjectivity problems which can be solved by localization and the Proj-functor. The following consequence of Runge's approximation theorem is fundamental for the classical applications in 3.4.1 – 3.4.3.

For an open set $\Omega \subseteq \mathbb{C}$ (or $\Omega \subseteq \mathbb{R}^d$) we call a sequence $(\Omega_n)_{n\in\mathbb{N}}$ an open and relatively compact exhaustion of $\Omega$ if each $\Omega_n$ is open and relatively compact, $\overline{\Omega}_n \subset \Omega_{n+1}$, and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. Then every compact subset of $\Omega$ is contained in some $\Omega_n$. The space of holomorohic functions on $\Omega$ is denoted by $\mathscr{H}(\Omega)$.

**Theorem 3.4.1** *Let $(\Omega_n)_{n\in\mathbb{N}}$ be an open and relatively compact exhaustion of $\Omega \subseteq \mathbb{C}$. Then the spectrum $\mathscr{X}$ consisting of the spaces $X_n = \mathscr{H}(\Omega_n)$ and the restriction maps $\varrho_m^n$ satisfies* $\operatorname{Proj} \mathscr{X} \cong \mathscr{H}(\Omega)$ *and* $\operatorname{Proj}^1 \mathscr{X} = 0$.

**Proof.** For each $n \in \mathbb{N}$ choose a compact set $K_n \subset \Omega$ with $\overline{\Omega}_n \subset K_n$ such that each component $A_i$ of the complement of $K_n$ taken in the extended complex plane contains a point $\alpha_i$ in the complement of $\Omega$. We fix $m \geq n$ such that $K_n \subset \Omega_m$. If $f \in \mathscr{H}(\Omega_m)$ and $\varepsilon > 0$ are given there is a rational function $r$ with poles only in $\{\alpha_i\}$ (in particular, $r \in \mathscr{H}(\Omega)$) such that $\|f - r\|_{K_n} = \sup_{z\in K_n} |f(z) - r(z)| \leq \varepsilon$.

We endow $\mathscr{H}(\Omega_n)$ with its usual Fréchet space topology of uniform convergence on compact subsets. For any $U \in \mathscr{U}_0(\mathscr{H}(\Omega_n))$ there are a compact set $M \subset \Omega_n \subset K_n$ and $\varepsilon > 0$ such that $\{f \in \mathscr{H}(\Omega_n) : \|f\|_M \leq \varepsilon\} \subseteq U$. Hence, the consequence of Runge's theorem given above shows $\varrho_m^n(X_m) \subseteq \varrho^n(\operatorname{Proj} \mathscr{X}) + U$. Theorem 3.2.1 implies $\operatorname{Proj}^1 \mathscr{X} = 0$. □

Since in the above proof the spectrum $X_n$ is equivalent to the spectrum consisting of the Banach spaces $\mathscr{H}^\infty(\Omega_n)$ of bounded holomorphic functions on $\Omega_n$ it is no surprise that $m \geq n$ only depends on $n$ and not on the 0-neighbourhood $U_n$.

### 3.4.1 The Mittag-Leffler theorem

If $\Omega$ is an open subset of $\mathbb{C}$, $(z_k)_{k\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ a sequence without accumulation point in $\Omega$ and $p_k(z) = \sum_{l=1}^{m_k} \frac{c_{l,k}}{(z-z_k)^l}$ a sequence of principal parts there is a meromorphic function in $\Omega$ with poles only in $\{z_k : k \in \mathbb{N}\}$ and having the principal parts $p_k$ in $z_k$.

This problem is localized by choosing an open and relatively compact exhaustion $\Omega_n$ of $\Omega_n$. With $X_n = H(\Omega_n)$, $Y_n$ the space of meromorphic functions on $\Omega_n$ with poles only in $\{z_k : k \in \mathbb{N}\}$ (there are only finitely many $z_k$ in $\Omega_n$ since this set is relatively compact and $(z_k)_{k\in\mathbb{N}}$ has no accumulation point) and $Z_n$ the image of the assignment of the principal parts to a function in $Y_n$ we obtain with the restictions as spectral mappings an exact sequence $0 \to \mathscr{X} \to \mathscr{Y} \to \mathscr{Z} \to 0$ of projective spectra.

Since $\operatorname{Proj}^1 \mathscr{X} = 0$ we obtain that $0 \to \operatorname{Proj} \mathscr{X} \to \operatorname{Proj} \mathscr{Y} \to \operatorname{Proj} \mathscr{Z} \to 0$ is exact which is the assertion of the Mittag-Leffler theorem.

### 3.4.2 Separating singularities

Let $\Omega_1$ and $\Omega_2$ be open subsets of $\mathbb{C}$ and $\varphi \in H(\Omega)$ with $\Omega = \Omega_1 \cap \Omega_2$. Then there are $\varphi_j \in H(\Omega_j)$ with $\varphi = \varphi_1|_\Omega - \varphi_2|_\Omega$.

To localize this problem we choose open and relatively compact exhaustions $\Omega_{j,n}$ of $\Omega_j$ and set $Z_n = H(\tilde{\Omega}_n)$ with $\tilde{\Omega}_n = \Omega_{1,n} \cap \Omega_{2,n}$, $Y_n = H(\Omega_{1,n}) \times H(\Omega_{2,n})$ and $X_n = H(\Omega_{1,n} \cup \Omega_{2,n})$. The spectral maps are again the (the product of the) restrictions. With $f_n : X_n \to Y_n$, $\varphi \mapsto (\varphi|_{\Omega_{1,n}}, \varphi|_{\Omega_{2,n}})$ and $g_n : Y_n \to Z_n$, $(\varphi_1, \varphi_2) \mapsto \varphi_1|_{\tilde{\Omega}_n} - \varphi_2|_{\tilde{\Omega}_n}$ we obtain a complex $0 \to \mathscr{X} \to \mathscr{Y} \to \mathscr{Z} \to 0$ which is exact at $\mathscr{X}$ and $\mathscr{Y}$. The proof of the exactness at $\mathscr{Z}$ is as hard as the original problem but in view of 3.1.8 we only need $\tau^n_{n+1}(Z_{n+1}) \subseteq g_n(Y_n)$, i.e. $\varphi|_{\tilde{\Omega}_n} = \varphi_1|_{\tilde{\Omega}_n} - \varphi_2|_{\tilde{\Omega}_n}$ for $\varphi \in H(\tilde{\Omega}_{n+1})$ and this can be seen easily with the aid of Cauchy's integral formula. Indeed, since $\overline{\Omega}_{1,n} \cap \overline{\Omega}_{2,n}$ is compact in $\tilde{\Omega}_{n+1}$ there is a cycle $\Gamma$ in $\tilde{\Omega}_{n+1} \setminus \overline{\Omega}_{1,n} \cap \overline{\Omega}_{2,n}$ such that $\varphi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\zeta)}{\zeta - z} d\zeta$ holds for $z \in \overline{\Omega}_{1,n} \cap \overline{\Omega}_{2,n}$ and $\varphi \in H(\tilde{\Omega}_{n+1})$. If we decompose $\Gamma = \Gamma_1 - \Gamma_2$ with cycles $\Gamma_j$ in $\tilde{\Omega}_{n+1} \setminus \overline{\Omega}_{n,j}$ we can solve this local problem with $\varphi_j(z) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{\varphi(\zeta)}{\zeta - z} d\zeta$ for $z \in \Omega_{j,n}$.

As in the previous situation, the kenel spectrum $\mathscr{X}$ satisfies $\operatorname{Proj}^1 \mathscr{X} = 0$ and we thus obtain the desired result. Usually, this special instance of the additive Cousin problem is solved using a $\mathscr{C}^\infty$-partition of unity to produce a decomposition with $\mathscr{C}^\infty$-functions and then one gets a holomorphic solution with the aid of the next example.

### 3.4.3 Surjectivity of $\overline{\partial}$

We prove that $\overline{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})$ is surjective on $\mathscr{C}^\infty(\Omega)$ for each open set $\Omega \subseteq \mathbb{C}$.

This is obtained in a similar way. We choose an exhaustion as in the first example and set $X_n = H(\Omega_n)$ and $Y_n = Z_n = \mathscr{C}^\infty(\Omega_n)$. The spectral maps are once more the restrictions, $f_n : X_n \to Y_n$ is the embedding and $g_n = \overline{\partial} : Y_n \to Z_n$. By 3.1.8 we have to solve $\overline{\partial} g = f$ in $\Omega_n$ for a function $f \in \mathscr{C}^\infty(\Omega_{n+1})$ and this can be done by choosing a $\mathscr{C}^\infty$-function $\varphi$ with support in $\Omega_{n+1}$ which takes the value 1 on $\overline{\Omega}_n$ and setting $g(z) = \frac{-1}{\pi} \int_\mathbb{C} \frac{(f\varphi)(\zeta)}{\zeta - z} d\lambda^2(\zeta)$ where $\lambda^2$ is the two-dimensional Lebesque measure on $\mathbb{C} \cong \mathbb{R}^2$.

Again we use $\operatorname{Proj}^1 \mathscr{X} = 0$ to obtain the desired surjectivity. The method of this example carries over to general partial differential operators with constant coefficients:

### 3.4.4 Surjectivity of $P(D)$ on $\mathscr{C}^\infty(\Omega)$

Let $\Omega$ be an open subset of $\mathbb{R}^d$, $P$ a non-zero polynomial in $d$ variables and $D = (-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_d})$. Surjectivity of $P(D) : \mathscr{C}^\infty(\Omega) \to \mathscr{C}^\infty(\Omega)$ has been charaterized by Malgrange [44].

The localization of the problem is obtained precisely as for the $\overline{\partial}$-operator by multiplying $f \in \mathscr{C}^\infty(\Omega_{n+1})$ with a cut-off function and forming the convolution with a fundamental solution $E$ of the operator. The difference to the previous examples is that the kernel spectrum formed by the spaces $X_n = \{f \in \mathscr{C}^\infty(\Omega_n) : P(D)f = 0\}$ need not automatically satisfy $\operatorname{Proj}^1 \mathscr{X} = 0$. From the characterization in theorem 3.2.8 we get $\operatorname{Proj}^1 \mathscr{Y} = 0$ where $\mathscr{Y}$ consists of the spaces $\mathscr{C}^\infty(\Omega_n)$ and the restrictions and using the exact sequence

$$0 \to \operatorname{Proj} \mathscr{X} \to \operatorname{Proj} \mathscr{Y} \to \operatorname{Proj} \mathscr{Z} \to \operatorname{Proj}^1 \mathscr{X} \to \operatorname{Proj}^1 \mathscr{Y}$$

from corollary 3.1.5 we deduce that surjectivity of $P(D)$ on $\mathscr{C}^\infty(\Omega)$ is equivalent to $\operatorname{Proj}^1 \mathscr{X} = 0$.

We claim that the condition of 3.2.1 (which by 3.2.8 characterizes the vanishing of $\operatorname{Proj}^1 \mathscr{X}$) holds if and only if $\Omega$ is $P$-convex for supports, i.e. for each compact set $K \subset \Omega$ and each $p \in \mathbb{N}$ there is another compact set $K' \subset \Omega$ such that each $\mu \in \mathscr{E}'(\Omega) = \mathscr{C}^\infty(\Omega)'$ with $P(-D)\mu$ having order less than $p$ and support in $K$ satisfies $\operatorname{supp} \mu \subseteq K'$.

Indeed, given $n \in \mathbb{N}$ and $U \in \mathscr{U}_0(X_n)$ there are a compact set $K$ and $p \in \mathbb{N}$ such that $U$ contains $\{f \in \mathscr{C}^\infty(\Omega_n) : \sum_{|\alpha| \le p} \|D^\alpha f\|_K < \varepsilon\}$ for some $\varepsilon > 0$. We choose $m > n$ such that $K' \subset \Omega_m$ and any $k > m$. Given $\nu \in (\varrho^n_k(X_k) + U)^\circ$ the distribution $\mu = \check{E} * \nu \in \mathscr{D}'(\mathbb{R}^d)$ has support in $\overline{\Omega_k}$ (since for a test function $\varphi$ with $\operatorname{supp} \varphi \cap \overline{\Omega_k} = \emptyset$ we have $\mu(\varphi) = \nu * \check{E}(\varphi) = \nu(E * \varphi) = 0$ as $E * \varphi \in X_k$ and $\nu \in \varrho^n_k(X_k)^\circ$). In particular, $\mu \in \mathscr{E}'(\Omega)$ and $P$-convexity implies $\operatorname{supp} \mu \in \Omega_n$ which yields $\nu(f) = P(-D)\mu(f) = \mu(P(D)f) = 0$ for $f \in \varrho^n_m(X_m)$, hence $\nu \in \varrho^n_m(X_m)^\circ$ and the theorem of bipolars gives

$$\varrho^n_m(X_m) \subseteq \overline{\varrho^n_k(X_k) + U} \subseteq \varrho^n_k(X_k) + 2U.$$

In the same way it is proved that this condition implies $P$-convexity.

By regularization one can show that the formulation of $P$-convexity given here is equivalent to the usual formulation ($\operatorname{supp} P(-D)\varphi \subseteq K$ implies $\operatorname{supp} \varphi \subseteq K'$ for all test functions $\varphi \in \mathscr{D}(\Omega)$) as e.g. in Hörmander's book [34, §10.6].

### 3.4.5 Surjectivity of $P(D)$ on $\mathscr{D}'(\Omega)$

An open subset $\Omega$ of $\mathbb{R}^d$ is $P$-convex for singular supports if for each compact set $K \subset \Omega$ there is a another compact set $K' \subset \Omega$ such that $\operatorname{sing\,supp} P(-D)\mu \subset K$ implies $\operatorname{sing\,supp} \mu \subset K'$ for each $\mu \in \mathscr{E}'(\Omega)$ (where $\operatorname{sing\,supp} \mu$ is the singular support of a distribution $\mu$, i.e. the complement of the largest open set in which $\mu$ is a $\mathscr{C}^\infty$-function, see [35, §2.2]). We will now present a Proj$^1$-proof (similar to [49]) of the following result due to Hörmander [33] and [34, §10.7]:

> If $\Omega$ is $P$-convex for supports as well as for singular supports then $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is surjective.

We choose an open and relatively compact exhaustion $\Omega_n$ such that $K' \subset \Omega_{n+1}$ for $K = \overline{\Omega}_n$ in both definitions of $P$-convexity. The local problems $u = P(D)v$ in $\Omega_n$ for $u \in \mathscr{D}'(\overline{\Omega}_{n+1})$ are again solved by multiplying with a cut-off function and using a fundamental solution. We set $X_n = \mathscr{D}'_P(\overline{\Omega}_n) = \{u \in \mathscr{D}'(\overline{\Omega}_n) : P(D)u = 0\}$ and denote by $Z_n$ the Banach space $\{u \in L_2(\overline{\Omega}_n) : P(D)u = 0\} \subset X_n$ where $L_2(\overline{\Omega}_n) = \{u \in L_2(\mathbb{R}^d) : \operatorname{supp} u \subset \overline{\Omega}_n\}$. We want to show

$$(\star) \quad \varrho^n_m(X_m) \subseteq \varrho^n_k(X_k) + Z_n$$

for all $k \geq m = n + 1$ where $\varrho^n_m$ denotes the restriction map. Let us first note that $L_j = P(-D)(\mathscr{D}(\Omega)) \cap \mathscr{D}(\overline{\Omega}_j)$ is closed in $\mathscr{D}(\overline{\Omega}_j)$ since $\Omega$ is $P$-convex for supports (in fact, $L_j = \{\varphi \in \mathscr{D}(\overline{\Omega}_j) : \operatorname{supp} E * \varphi \subset \Omega_{j+1}\}$ where $E$ is a fundamental solution of $P(-D)$) and for the same reason $M_m = P(-D)(\mathscr{E}'(\Omega)) \cap F_m$ is closed in $F_m = \{u \in L_2(\overline{\Omega}_m) : u \in \mathscr{C}^\infty(\Omega \setminus \overline{\Omega}_{m-1})\}$ which is endowed with the initial topology with respect to the inclusion into $L_2(\overline{\Omega}_m)$ and the restriction map $F_m \to \mathscr{C}^\infty(\Omega \setminus \overline{\Omega}_{m-1})$. The map

$$\sigma = (\sigma_1, \sigma_2) : \ \mathscr{D}(\overline{\Omega}_m)/L_m \longrightarrow \mathscr{D}(\overline{\Omega}_k)/L_k \times F_m/M_m$$

$$\varphi + L_m \mapsto (\varphi + L_k, \varphi + M_m)$$

is continuous and injective (since $\mathscr{D}(\overline{\Omega}_m) \cap L_k = L_m$) and we will prove that it has closed range. Indeed, if $(\varphi + L_k, u + M_m)$ is in the closure of $\operatorname{im}\sigma$ then $\varphi - u \in L_2(\overline{\Omega}_m) \cap P(-D)(\mathscr{E}'(\Omega))$ (the subspace of the product described by this condition is closed and contains $\operatorname{im}\sigma$) hence there is $\mu \in \mathscr{E}'(\Omega)$ with $P(-D)\mu = \varphi - u$. Since $u$ is smooth outside $\overline{\Omega}_{m-1}$ we have $\operatorname{sing\,supp} P(-D)\mu \subset \overline{\Omega}_{m-1}$ hence $\operatorname{sing\,supp}\mu$ is a compact subset of $\Omega_m$ by $P$-convexity for singular supports. Moreover, $P(-D)\mu \in L_2(\overline{\Omega}_m)$ implies that $\mu$ is locally an $L_2$-function (since $P(-D)$ has a regular fundamental solution [34, theorem 10.2.1]) which has compact support in $\Omega$ (since $\Omega$ is $P$-convex). We can thus decompose $\mu = \psi - v$ with $\psi \in \mathscr{D}(\Omega)$ and $v \in L_2(\overline{\Omega}_m)$ ($v$ is obtained by multiplying $u$ with a function in $\mathscr{D}(\overline{\Omega}_m)$ which 1 on a neighbourhood of $\operatorname{sing\,supp}\mu$) and obtain that

$$\chi = \varphi - P(-D)\psi = u - P(-D)v \in \mathscr{D}(\Omega)$$

has support in $\overline{\Omega}_m$. This gives $\chi \in \mathscr{D}(\overline{\Omega}_m)$ and $\sigma(\chi + L_m) = (\varphi + L_k, u + M_m)$ which proves that $\operatorname{im}\sigma$ is closed.

Now we can prove ($\star$). Any $u \in \mathscr{D}'_P(\overline{\Omega}_m)$ vanishes on $L_m = \mathscr{D}(\overline{\Omega}_m) \cap P(-D)\mathscr{D}(\Omega)$ (since $u(P(-D)\varphi) = P(D)u(\varphi)$) and thus induces a continuous linear functional $\tilde{u}$ on $\mathscr{D}(\overline{\Omega}_m)/L_m$. The open mapping theorem shows that $\sigma$

is a monohomomorphism and the Hahn-Banach theorem implies that there are continuous linear functionals $v$ on $\mathscr{D}(\overline{\Omega}_k)$ vanishing on $L_k$ and $w$ on $F_m$ vansihing on $M_m$ with $\tilde{u} = \tilde{v}\circ\sigma_1 + \tilde{w}\circ\sigma_2$ (where $\tilde{v}$ and $\tilde{w}$ are the corresponding functionals on the quotient spaces). Since $v$ vanishes on $L_k$ it belongs to $\mathscr{D}'_P(\overline{\Omega}_k)$ and since $L_2(\overline{\Omega}_{m-1}) \subset F_m$ we obtain $w \in L_2(\overline{\Omega}_{m-1})' = L_2(\overline{\Omega}_{m-1})$. This proves $\varrho^m_{m-1}(X_m) \subseteq \varrho^k_{m-1}(X_k) + L_2(\overline{\Omega}_{m-1})$ and therefore also $(\star)$.

By the remark after 3.2.17 condition $(\star)$ implies that the spectrum $\mathscr{X} = (X_n, \varrho^m_n)$ satisfies condition $(\mathrm{P}_3)$ defined in 3.2.17. Because of theorem 3.2.18 we still need that $\mathscr{X}$ is reduced to conclude $\mathrm{Proj}^1 \mathscr{X} = 0$ and hence the surjectivity of $P(D)$ on $\mathscr{D}'(\Omega)$. But reducedness follows again from the Hahn-Banach theorem and the injectivity of $\sigma_1 : \mathscr{D}(\overline{\Omega}_m)/L_m \to \mathscr{D}(\overline{\Omega}_k)/L_k$ and the proof is thus complete.

Hörmander's proof used $P$-convexity for singular supports in a rather intricate way to construct explicitely a continuous seminorm on $\mathscr{D}(\Omega)$ of the form $p(\varphi) = \sum_\alpha \sup |g_\alpha D^\alpha \varphi|$ to which the Hahn-Banach theorem applies.

It is shown in [34, 10.6.6 and 10.7.6] that the $P$-convexity conditions are also necessary for the surjectivity of $P(D)$ on $\mathscr{D}'(\Omega)$. Let us also note that by [35, 7.3.9] open convex sets are $P$-convex for supports as well as for singular supports. The surjectivity of $P(D)$ on $\mathscr{D}'(\Omega)$ for convex $\Omega$ can also be proved as in the next application.

### 3.4.6 Differential operators for ultradifferentiable functions of Roumieu type

We will now explain how theorem 3.2.18 and its variant corollary 3.3.11 simplify considerably arguments of Braun, Meise, and Vogt [16, 18], Braun [14], and Langenbruch [41] used for the characterization of surjective partial differential operators on spaces of Roumieu type (sufficiency of conditions $(P_2)$ and $(P_2^\star)$ was only proved in 1996 [69] after the above mentionned articles appeared).

To define ultradifferentiable functions we follow Braun, Meise, and Taylor [15] and call an increasing continuous function $\omega : [0,\infty) \to [0,\infty)$ a weight function if for some constant $C$

$$(\alpha)\ \omega(2t) \le C(1+\omega(t)), \qquad (\beta)\ \int_0^\infty \frac{\omega(t)}{1+t^2}dt < \infty,$$

$$(\gamma)\ \lim_{t\to\infty} \frac{\log t}{\omega(t)} = 0, \text{ and} \qquad (\delta)\ \varphi(t) = \omega(\exp(t)) \text{ is convex.}$$

$\omega$ is extended to $\mathbb{C}^d$ by $\omega(z) = \omega(|z|)$ (where $|z|$ is the euclidean norm). The Young conjugate $\varphi^* : [0,\infty) \to \mathbb{R}$ of $\varphi$ is defined by

$$\varphi^*(y) = \sup_{x\ge 0} xy - \varphi(x).$$

For a compact set $K \subset \mathbb{R}^d$ we set

$$J(K) = \left\{ f = (f_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \mathscr{C}(K)^{\mathbb{N}_0^d} : f_\alpha|_{\mathring{K}} \in \mathscr{C}^\infty(\mathring{K}),\ f_\alpha|_{\mathring{K}} = \partial^\alpha f_0|_{\mathring{K}} \right\},$$

$$\|f\|_{K,N} = \sup_{\alpha\in\mathbb{N}_0^d} \sup_{x\in K} |f_\alpha(x)| \exp(-\varphi^*(|\alpha|N)/N) \text{ for } f \in J(K),$$

(here $|\alpha| = \alpha_1 + \ldots + \alpha_d$ denotes the length of the multi-index which hardly can be confused with the euclidean norm),

$$\mathscr{E}_{\{\omega\}}(K) = \{f \in J(K) : \exists\, N \in \mathbb{N} \quad \|f\|_{K,N} < \infty\},$$

and for an open set $\Omega \subset \mathbb{R}^d$ we finally get the space of ultradifferentiable functions on $\Omega$

$$\mathscr{E}_{\{\omega\}}(\Omega) = \left\{ f \in \mathscr{C}^\infty(\Omega) : (\partial^\alpha f|_K)_{\alpha\in\mathbb{N}_0^d} \in \mathscr{E}_{\{\omega\}}(K) \text{ for all } K \subset \Omega \text{ compact} \right\}.$$

In this definition it is clearly enough to consider a fixed compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of $\Omega$ (for which we always assume $K_n \subset \mathring{K}_{n+1}$ and that $K_n$ is the closure of $\mathring{K}_n$). If we denote the restriction maps $\mathscr{E}_{\{\omega\}}(K_m) \to \mathscr{E}_{\{\omega\}}(K_n)$ by $\varrho_m^n$ we obtain a projective spectrum $\mathscr{E}_{\{\omega\}}^{\Omega} = (\mathscr{E}_{\{\omega\}}(K_n), \varrho_m^n)$ whose projective limit is canonically isomorphic to $\mathscr{E}_{\{\omega\}}(\Omega)$.

The main example for this type of function spaces is obtained for $\omega(t) = t^\beta$ with $0 < \beta < 1$. The Young conjugate is then easily computed as $\varphi^*(y) = y/\beta \log(y/e\beta)$ and since $\exp(-\varphi^*(aN)/N)) = (N/\beta)^{-a/\beta}(a/e)^{-a/\beta}$ we get with $p = 1/\beta$ from Stirling's formula

$$\mathscr{E}_{\{\omega\}}(\Omega) = \left\{ f \in \mathscr{C}^\infty(\Omega) : \forall\, n \in \mathbb{N}\ \exists\, N \in \mathbb{N} \sup_{\alpha\in\mathbb{N}_0^d} \sup_{x\in K_n} \frac{|\partial^\alpha f(x)|}{N^{|\alpha|}(\alpha!)^p} < \infty \right\}.$$

We thus obtain the classical Gevrey-classes of exponent $p$ which are usually denoted by $\Gamma^{\{p\}}(\Omega)$. Note that the case $p = 1$ (for which we would obtain the space of real analytic functions on $\Omega$) is not covered by this setting because of condition ($\beta$). Besides this formal reason the crucial property which excludes spaces of real analytic functions is that $\mathscr{E}_{\{\omega\}}(\Omega)$ contains many functions with compact support as is shown in [15], hence $\mathscr{E}_{\{\omega\}}(\Omega)$ is non-quasianalytic.

Let now $P(z) = \sum_{|\delta|\le k} c_\delta z_1^{\delta_1} \cdots z_d^{\delta_d}$ be a non-constant polynomial in $d$ variables. It is shown in [15] that $P(D)$ acts continuously on $\mathscr{E}_{\{\omega\}}(\Omega)$ and also that the formal differential operator

$$P(D) : (f_\alpha)_{\alpha\in\mathbb{N}_0^d} \mapsto \Big( \sum_{|\delta|\le k} c_\delta (-i)^{|\delta|} f_{\alpha+\delta} \Big)_{\alpha\in\mathbb{N}_0^d}$$

is a continuous linear map on $\mathscr{E}_{\{\omega\}}(K)$. If $\mathscr{N}(\omega, P, \Omega)$ is the spectrum consisting of the kernels $N(\omega, P, K_n) = \{f \in \mathscr{E}_{\{\omega\}}(K_n) : P(D)f = 0\}$ we obtain a complex

$$0 \to \mathscr{N}(\omega, P, \Omega) \longrightarrow \mathscr{E}_{\{\omega\}}^{\Omega} \longrightarrow \mathscr{E}_{\{\omega\}}^{\Omega}$$

of projective spectra whose projective limit is

$$0 \to \ker P(D) \longrightarrow \mathscr{E}_{\{\omega\}}(\Omega) \stackrel{P(D)}{\longrightarrow} \mathscr{E}_{\{\omega\}}(\Omega).$$

Using compactly supported functions in $\mathscr{E}_{\{\omega\}}(\Omega)$ we get $\varrho^n_{n+1}(\mathscr{E}_{\{\omega\}}(K_{n+1})) \subseteq \varrho^n(\mathscr{E}_{\{\omega\}}(\Omega))$ which implies $\mathrm{Proj}^1 \mathscr{E}^{\Omega}_{\{\omega\}} = 0$ (e.g. by theorem 3.2.1 applied to the discrete topologies). The assumptions of proposition 3.1.8 are again verified using a fundamental solution. We therefore obtain that $P(D)$ is surjective on $\mathscr{E}_{\{\omega\}}(\Omega)$ if and only if $\mathrm{Proj}^1 \mathscr{N}(\omega, P, \Omega) = 0$.

To obtain explicit conditions from this characterization let us first note that by [11] $\mathscr{E}_{\{\omega\}}(K_n)$ and thus also the closed subspaces $N(\omega, P, K_n)$ are (LS)-spaces. The dual condition $(P_3^*)$ in corollary 3.3.12 can be evaluated directly by using Fourier transformation if $\Omega$ is convex. In this case we can of course assume that the exhaustion $(K_n)_{n\in\mathbb{N}}$ consists of convex sets, too. We then have the support functionals

$$h_n(y) = h_{K_n}(y) = \sup_{x \in K_n} \langle x, y \rangle.$$

Bonet, Meise, and Taylor [11] proved a Paley-Wiener-Schwartz theorem for distributions in $\mathscr{E}'_{\{\omega\}}(\Omega)$, namely that the Fourier transform

$$\mathscr{F} : \mathscr{E}_{\{\omega\}}(K)'_\beta \to A_{\{\omega\}}(\mathbb{C}^d, K), \quad \mu \mapsto \mathscr{F}(\mu)(z) = \mu(\exp(-i\langle \cdot, z \rangle))$$

is an isomorphism between the strong dual of $\mathscr{E}_{\{\omega\}}(K)$ and a weighted space of entire functions.

For an algebraic variety V in $\mathbb{C}^d$ we denote by $A(V)$ the space of functions which locally have extensions to holomorphic functions on open sets in $\mathbb{C}^d$ and set

$$A_{\{\omega\}}(V, K) = \{f \in A(V) : \forall\, N \in \mathbb{N} \quad |f|_{K,N} < \infty\}$$

$$\text{where} \quad |f|_{K,N} = \sup_{z \in V} |f(z) \exp(-h_K(\mathrm{Im} z) - \omega(z)/N)|.$$

$A_{\{\omega\}}(V, K)$ is endowed with the topology given by the system of norms $|\cdot|_{K,N}$. The intuitive idea that $\mathscr{F}$ induces an isomorphism between $N(\omega, P, K)' = \mathscr{E}_{\{\omega\}}(K)'/N(\omega, P, K)^\perp$ and $A_{\{\omega\}}(V, K)$ with $V = \{z \in \mathbb{C}^d : P(-z) = 0\}$ is far from being obvious but it is shown by Braun, Meise and Vogt [18, propsition 2.4] that this is indeed true if $P$ is square-free (this assumption is no problem since the composition of commuting operators is surjective if so is each factor).

Having this description at hand we immediately get from corollary 3.3.12 the following characterization ($\mathscr{N}(\omega, P, \Omega)$ is a reduced spectrum because of the injectivity of the embedding $A_{\{\omega\}}(V, K_n) \hookrightarrow A_{\{\omega\}}(V, K_{n+1})$, hence $(P_3^*)$ characterizes $\mathrm{Proj}^1 \mathscr{N}(\omega, P, \Omega) = 0$ and thus the desired surjectivity):

**Theorem 3.4.2** *If $\Omega$ is convex and $P$ is square-free then $P(D)$ is surjective on $\mathscr{E}_{\{\omega\}}(\Omega)$ if and only if the variety $V = \{z \in \mathbb{C}^d : P(-z) = 0\}$ satisfies the following Phragmen-Lindelöf condition:*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\, N \in \mathbb{N}\ \forall\, M \in \mathbb{N}\ \exists\, K \in \mathbb{N},\ S > 0$$

*such that for each $f \in A(V)$ conditions (1) and (2) imply (3) where*

$$\begin{array}{ll} (1) & \log|f(z)| \leq h_n(\mathrm{Im} z) + \omega(z)/N \\ (2) & \log|f(z)| \leq h_k(\mathrm{Im} z) + \omega(z)/K \\ (3) & \log|f(z)| \leq h_m(\mathrm{Im} z) + \omega(z)/M + S. \end{array}$$

In [18] variants of this theorem could be proved by specialized hard analysis only for the case $\Omega = \mathbb{R}^d$ since the authors had to verify a condition $(P_1^*)$ of Vogt [62] (see example 3.2.19) which implies the assumption of the Retakh-Palamodov theorem 3.2.9 but which is neither necessary for $\mathrm{Proj}^1 \mathscr{N} = 0$ nor easy to check. Of course, to see whether for a concrete polynomial $P$ the Phragmen-Lindelöf condition holds one still needs further analytical tools but theorem 3.4.2 itself follws immediately from the abstract theory.

Several versions of 3.4.2 for convex $\Omega$ had been proved by Braun [13, 14] using sufficient conditions for $\mathrm{Proj}^1 \mathscr{N} = 0$ which were weaker than condition $(P_1^*)$ but still too strong to be necessary.

The surjectivity of $P(D)$ on $\mathscr{E}_{\{\omega\}}(\Omega)$ for non-convex $\Omega$ was characterized by Langenbruch [41] who also used $(P_2^*)$ as a necessary condition for the surjectivity. Because of corollary 3.3.12 one gets a first characterization which can be further evaluated as in [41].

Let us finally note that also convolution operators on classes $\mathscr{E}_{\{\omega\}}$ can be investigated with the aid of the present methods, see [17].

# 4

# Uncountable projective spectra

We have seen quite a number of useful results about countable projective limits in the previous chapter and it is a natural question to which extent these results carry over to arbitrary projective limits. Such projective limits are by no means artificial: every complete locally convex space is the projective limit of Banach spaces and such a representation is natural and useful.

However, we will see that the behaviour of the generalized projective limit functor is much worse than in the countable case. Even for very "good" projective spectra whose limits are for instance (LS)-spaces we can have $\mathrm{Proj}^1 \mathscr{X} \neq 0$.

Since the positive results are not very promising for the theory of locally convex spaces and in view of the existing literature (e.g. the lecture notes of C.U. Jensen [37]) our presentation will not be exhaustive. Our aim is to develop the theory to an extent which enables us to calculate $\mathrm{Proj}^k \mathscr{X}$ where $\mathscr{X}$ is a projective spectrum representing the space $\varphi$ of all finite sequences endowed with the strongest locally convex topology. These results will be used later.

## 4.1 Projective spectra of linear spaces

Throughout this chapter, $(I, \leq)$ will be a directed set, i.e. $\leq$ is a reflexive ($\alpha \leq \alpha$) and transitive ($\alpha \leq \beta$ and $\beta \leq \gamma$ imply $\alpha \leq \gamma$) relation on $I$ such that for all $\alpha, \beta \in I$ there exists $\gamma \in I$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$. Usually, it is required that in addition the order relation is antisymmetric, i.e. $\alpha \leq \beta$ and $\beta \leq \alpha$ imply $\alpha = \beta$. However, in connection with the representation of locally convex spaces the following example is natural. Let $X$ be a locally convex space and $I = cs(X)$ the system of all continuous seminorms on $X$. Together with the order $p \leq q$ if the identical map $(X, p) \longrightarrow (X, q)$ is continuous, i.e. there is $C > 0$ such that $p(x) \leq Cq(x)$ for all $x \in X$, $(I, \leq)$ becomes a directed set.

J. Wengenroth: LNM 1810, pp. 59–76, 2003.

Of course, the difference to the usual definition is not really essential. If we define an equivalence relation $\alpha \sim \beta$ if $\alpha \leq \beta$ and $\beta \leq \alpha$ and denote by $[\alpha]$ the equivalence class corresponding to $\alpha \in I$ then $[\alpha] \prec [\gamma]$ if $\alpha \leq \gamma$ is a well-defined order which is antisymmetric and such that $(I/\sim, \prec)$ is a directed set.

**Definition 4.1.1** *A projective $I$-spectrum consists of linear spaces $X_\alpha$ for $\alpha \in I$, and linear maps $\varrho^\alpha_\beta$ for $\alpha \leq \beta$ such that $\varrho^\alpha_\alpha = id_{X_\alpha}$ and $\varrho^\alpha_\beta \circ \varrho^\beta_\gamma = \varrho^\alpha_\gamma$ for $\alpha \leq \beta \leq \gamma$. If $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ and $\mathscr{Y} = (Y_\alpha, \sigma^\alpha_\beta)$ are two projective $I$-spectra, a morphism $f = (f_\alpha)_{\alpha \in I} : \mathscr{X} \longrightarrow \mathscr{Y}$ consists of linear maps $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ such that $f_\alpha \circ \varrho^\alpha_\beta = \sigma^\alpha_\beta \circ f_\beta$ for all $\alpha \leq \beta$. For a projective $I$-spectrum $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ we set*

$$\operatorname{Proj} \mathscr{X} = \Big\{(x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha : \varrho^\alpha_\beta(x_\beta) = x_\alpha \text{ for } \alpha \leq \beta\Big\}$$

*and denote by $\varrho^\alpha : \operatorname{Proj} \mathscr{X} \longrightarrow X_\alpha$ the canonical projection. If $f = (f_\alpha)_{\alpha \in I} : \mathscr{X} \longrightarrow \mathscr{Y}$ is a morphism between projective $I$-spectra we define* $\operatorname{Proj}(f) : \operatorname{Proj} \mathscr{X} \longrightarrow \operatorname{Proj} \mathscr{Y}$ *by $(x_\alpha)_{\alpha \in I} \mapsto (f_\alpha(x_\alpha))_{\alpha \in I}$.*

This is the usual definition of projective spectra. However, in view of the example above one would like to have a category of spectra where the index sets may vary. If $X$ and $Y$ are complete locally convex spaces and $f : X \longrightarrow Y$ is a continuous linear map there should be a morphism $\mathscr{X} \longrightarrow \mathscr{Y}$ where $\mathscr{X}$ is the spectrum consisting of the spaces $X_p = \widetilde{(X, p)}$, $p \in cs(X)$ and $\mathscr{Y}$ is the spectrum corresponding to $Y$. To realize this in our setting one would have to change the natural index sets $cs(X)$ and $cs(Y)$, e.g. we could take $I = \{(p, q) \in cs(X) \times cs(Y) : f : (X, p) \longrightarrow (Y, q) \text{ continuous}\}$. Then we can indeed find a morphism $(f_\alpha)_{\alpha \in I}$ between spectra such that $f = \operatorname{Proj}((f_\alpha)_{\alpha \in I})$. A more systematic way would be to mimic the definitions of Palamodov for the countable case. Some steps in this direction are done in 4.1.6 and 4.1.7 below. But as we said above, the results even for the simple case where only one index set is used are rather disappointing, and since there is no hope for the more complicated theory to allow better results there seems to be no justification yet to invent such a technical machinery.

We thus confine ourselves to the setting of 4.1.1. The abandonment of antisymmetry does not cause anything unexpected, if $\alpha \leq \beta$ and $\beta \leq \alpha$ then $X_\alpha$ and $X_\beta$ are isomorphic and $\varrho^\beta_\alpha$ and $\varrho^\alpha_\beta$ are inverse to each other.

**Proposition 4.1.2** *The class of projective $I$-spectra and morphisms is an abelian category which has sufficiently many injective objects, and* Proj *is an injective additive functor.*

**Proof.** The proof is almost the same as in the countable case, the kernel (cokernel) of a morphism is the spectrum of the kernels (cokernels) of the components, every morphism is a homomorphism, and for an $I$-spectrum

$\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ the associated free spectrum $\mathscr{F} = (\prod_{\gamma\leq\alpha} X_\gamma, \pi^\alpha_\beta)$, where $\pi^\alpha_\beta$ are the canonical projections, is an injective object. Indeed, if $f = (f_\alpha)_{\alpha\in I} : \mathscr{Y} = (Y_\alpha, \sigma^\alpha_\beta) \longrightarrow \mathscr{F}$ is a morphism and $h : \mathscr{Y} \longrightarrow \mathscr{Z} = (Z_\alpha, \tau^\alpha_\beta)$ is a monomorphism we define $\tilde{f} = (\tilde{f}_\alpha)_{\alpha\in I} : \mathscr{Z} \longrightarrow \mathscr{F}$ by $\tilde{f}_\alpha = \prod_{\gamma\leq\alpha} \widehat{f}_\gamma \circ \tau^\gamma_\alpha : Z_\alpha \longrightarrow \prod_{\gamma\leq\alpha} X_\gamma$, where $\widehat{f}_\gamma : Z_\gamma \longrightarrow X_\gamma$ is an extension of $pr_\gamma \circ f_\gamma : Z_\gamma \longrightarrow X_\gamma$ and $pr_\gamma : \prod_{\delta\leq\gamma} X_\delta \longrightarrow X_\gamma$ is the projection. It is then easy to check that $\tilde{f}$ is a morphism with $\tilde{f} \circ h = f$. Hence $\mathscr{F}$ is an injective object and $i = (i_\alpha)_{\alpha\in I} : \mathscr{X} \longrightarrow \mathscr{F}$ defined by $i_\alpha(x) = (\varrho^\gamma_\alpha(x))_{\gamma\leq\alpha}$ is a monomorphism. Finally, Proj is easily seen to be injective. □

Now, the derived functors $\mathrm{Proj}^k$ can be defined using injective resolutions as in the countable case. Let us construct a particular resolution using a standard trick from homological algebra. Given $k \in \mathbb{N}_0$ and $\alpha \in I$ we set $I_{k,\alpha} = \{(\alpha_0, \ldots, \alpha_k) \in I^{k+1} : \alpha_0 \leq \ldots \leq \alpha_k \leq \alpha\}$, $F_{k,\alpha} = \prod_{(\alpha_0,\ldots,\alpha_k)\in I_{k,\alpha}} X_{\alpha_0}$, and denote by $\pi^\alpha_{k,\beta} : F_{k,\beta} \longrightarrow F_{k,\alpha}$ the canonical projection. Then $\mathscr{F}_k = (F_{k,\alpha}, \pi^\alpha_{k,\beta})$ are free projective $I$-spectra, hence injective. Moreover, we define $i_{k,\alpha} : F_{k,\alpha} \longrightarrow F_{k+1,\alpha}$ by $(x_{\alpha_0,\ldots,\alpha_k})_{(\alpha_0,\ldots,\alpha_k)\in I_{k,\alpha}} \longmapsto$

$$\left(\varrho^{\alpha_0}_{\alpha_1}(x_{\alpha_1,\ldots,\alpha_{k+1}}) + \sum_{j=1}^{k+1}(-1)^j x_{\alpha_0,\ldots,\widehat{\alpha}_j,\ldots,\alpha_{k+1}}\right)_{(\alpha_0,\ldots,\alpha_{k+1})\in I_{k+1,\alpha}}$$

where $(\alpha_0, \ldots, \widehat{\alpha}_j, \ldots, \alpha_{k+1}) = (\alpha_0, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{k+1})$. To shorten the formulas we will omit the spectral map and write $\sum_{j=0}^{k+1} (-1)^j x_{\alpha_0,\ldots,\widehat{\alpha}_j,\ldots,\alpha_{k+1}}$ for the expression above. For $x = (x_{\alpha_0,\ldots,\alpha_k})_{(\alpha_0,\ldots,\alpha_k)\in I_{k,\alpha}}$ we have

$$i_{k+1,\alpha} \circ i_{k,\alpha}(x) = \sum_{l=0}^{k+2} \sum_{j=0}^{k+1} z_{l,j} = 0$$

since the summands $z_{l,j}$ are of the form $\pm x_{\alpha_a,\ldots,\widehat{\alpha}_r,\ldots,\widehat{\alpha}_s,\ldots,\alpha_{k+2}}$ and each of these terms appears twice with different signs. If on the other hand $x = (x_{\alpha_0,\ldots,\alpha_k})_{(\alpha_0,\ldots,\alpha_k)\in I_{k,\alpha}}$ is in the kernel of $i_{k,\alpha}$ for some $k \in \mathbb{N}$ we set

$$y_{\alpha_0,\ldots,\alpha_{k-1}} = \begin{cases} (-1)^k x_{\alpha_0,\ldots,\alpha_{k-1},\alpha} & \text{if } \alpha_{k-1} \neq \alpha \\ 0 & \text{if } \alpha_{k-1} = \alpha \end{cases}$$

Then $i_{k-1,\alpha}\left((y_{\alpha_0,\ldots,\alpha_{k-1}})_{(\alpha_0,\ldots,\alpha_{k-1})\in I_{k-1,\alpha}}\right) = x$ since for $\alpha_k \neq \alpha =: \alpha_{k+1}$ we have

$$\sum_{j=0}^{k}(-1)^j y_{\alpha_0,\dots,\widehat{\alpha}_j,\dots,\alpha_k} = (-1)^k \sum_{j=0}^{k}(-1)^j x_{\alpha_0,\dots,\widehat{\alpha}_j,\dots,\alpha_k,\alpha} =$$
$$= (-1)^k \left(\sum_{j=0}^{k+1}(-1)^j x_{\alpha_0,\dots,\widehat{\alpha}_j,\dots,\alpha_{k+1}} - (-1)^{k+1} x_{\alpha_0,\dots,\alpha_k}\right)$$
$= x_{\alpha_0,\dots,\alpha_k}$, and for $\alpha_k = \alpha$ we also have

$$\sum_{j=0}^{k}(-1)^j y_{\alpha_0,\dots,\widehat{\alpha}_j,\dots,\alpha_k} = (-1)^k y_{\alpha_1,\dots,\alpha_{k-1}} = x_{\alpha_0,\dots,\alpha_k}.$$

We proved, that the complex

$$0 \longrightarrow \mathscr{X} \xrightarrow{i} \mathscr{F}_0 \xrightarrow{i_0} \mathscr{F}_1 \xrightarrow{i_1} \mathscr{F}_2 \longrightarrow \dots$$

(where $i$ is the monomorphism from the proof of 4.1.2) is an injective resolution of $\mathscr{X}$. Applying Proj and using the obvious isomorphism $\operatorname{Proj}\mathscr{F}_k \cong \prod_{\alpha_0\leq\dots\leq\alpha_k} X_{\alpha_0}$ we obtain the following result (see e.g. [37, théorème 4.1]).

**Theorem 4.1.3** *Let* $\mathscr{X} = (x_\alpha, \varrho^\alpha_\beta)$ *be a projective* $I$*-spectrum. We set* $F_k = \prod_{\alpha_0\leq\dots\leq\alpha_k} X_{\alpha_0}$ *and define* $d_k : F_k \longrightarrow F_{k+1}$ *by* $(x_{\alpha_0,\dots,\alpha_k})_{\alpha_0\leq\dots\leq\alpha_k} \longmapsto$

$$\left(\varrho^{\alpha_0}_{\alpha_1}(x_{\alpha_1,\dots,\alpha_{k+1}}) + \sum_{j=1}^{k+1}(-1)^j x_{\alpha_0,\dots,\widehat{\alpha}_j,\dots\alpha_k}\right)_{\alpha_0\leq\dots\leq\alpha_{k+1}}.$$

*Then* $(\operatorname{Proj}^k\mathscr{X})_{k\in\mathbb{N}_0}$ *is the cohomology of the complex*

$$0 \longrightarrow F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} F_2 \longrightarrow \dots,$$

*i.e.* $\operatorname{Proj}^0\mathscr{X} \cong \ker(d_0) \cong \operatorname{Proj}\mathscr{X}$ *and* $\operatorname{Proj}^k\mathscr{X} \cong \ker(d_k)/\operatorname{im}(d_{k-1})$.

For our next example it is convenient to have a variant of this result. A minimum $\gamma \in I$ of $\alpha, \beta \in I$ is defined by the properties $\gamma \leq \alpha$, $\gamma \leq \beta$ and $\delta \leq \alpha$, $\delta \leq \beta$ implies $\delta \leq \gamma$.

**Theorem 4.1.4** *Let* $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ *be a projective* $I$*-spectrum with an antisymmetric directed set* $(I, \leq)$ *such that each pair* $(\alpha,\beta) \in I^2$ *has a minimum* $\alpha\wedge\beta \in I$*. We set* $G_k = \prod_{(\alpha_0,\dots,\alpha_k)\in I^{k+1}} X_{\alpha_0\wedge\dots\wedge\alpha_k}$ *and define* $d_k : G_k \longrightarrow G_{k+1}$ *by* $(x_{\alpha_0,\dots,\alpha_k})_{(\alpha_0,\dots,\alpha_k)\in I^{k+1}} \longmapsto$

$$\left(\sum_{j=0}^{k+1}(-1)^j \varrho^{\alpha_0\wedge\dots\wedge\alpha_{k+1}}_{\alpha_0\wedge\dots\wedge\widehat{\alpha}_j\wedge\dots\wedge\alpha_{k+1}}(x_{\alpha_0,\dots,\widehat{\alpha}_j,\dots,\alpha_{k+1}})\right)_{(\alpha_0,\dots,\alpha_{k+1})\in I^{k+2}}.$$

*Then* $(\operatorname{Proj}^k \mathscr{X})_{k\in\mathbb{N}_0}$ *is the cohomology of the complex*

$$0 \longrightarrow G_0 \xrightarrow{d_0} G_1 \xrightarrow{d_1} G_2 \longrightarrow \dots$$

**Proof.** The proof is almost the same as above. Since $\leq$ is antisymmetric the minimum of two elements is unique, $\alpha\wedge\beta = \beta\wedge\alpha$, and $(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$, thus the spaces $G_k$ and the "differentials" $d_k$ are well-defined. □

**Example 4.1.5** *Let* $X$ *be a paracompact topological space and let* $\mathscr{O}$ *be an open covering of* $X$ *which is stable with respect to finite unions and intersections. Then* $(\mathscr{O}, \subseteq)$ *is an antisymmetric directed set such that each pair* $U, V \in \mathscr{O}$ *has a minimum* $U \wedge V = U \cap V \in \mathscr{O}$. *Let* $X_U = \mathscr{C}(U)$ *be the space of continuous functions on* $U \in \mathscr{O}$ *and* $\varrho_V^U$ *be the restriction.*

*Then* $\mathscr{X} = (\mathscr{C}(U), \varrho_V^U)$ *is a projective* $\mathscr{O}$*-spectrum such that* $\operatorname{Proj}\mathscr{X} \cong C(X)$ *and* $\operatorname{Proj}^k \mathscr{X} = 0$ *for* $k \in \mathbb{N}$.

The first statement is quite obvious and we prove the second using 4.1.4. Let $f = (f_s)_{s\in\mathscr{O}^{k+1}} \in \prod_{(U_0,\dots,U_k)\in\mathscr{O}^{k+1}} \mathscr{C}(U_0\wedge\ldots\wedge U_k)$ with $d_k(f) = 0$ be given where $d_k$ is defined as in 4.1.4.

Let $(\varphi_t)_{t\in T}$ be a locally finite partition of unity subordinated to $\mathscr{O}$, i.e. each $\varphi_t$ is continuous with compact support contained in some $U_t \in \mathscr{O}$, each $x \in X$ has a neighbourhood on which all but finitely many $\varphi_t$ vanish, and $\sum_{t\in T} \varphi_t(x) = 1$ for all $x \in X$. Given $U_0, \dots, U_{k-1} \in \mathscr{O}$ we set

$$h_{U_0,\dots U_{k-1}} := \sum_{t\in T} \varphi_t f_{U_t,U_0,\dots,U_{k-1}},$$

more precisely, we define $v_t = \varphi_t f_{U_t,U_0,\dots,U_{k-1}}$ on $U_0 \cap \ldots \cap U_{k-1} \cap U_t$ and $v_t = 0$ on $U_0 \cap \ldots \cap U_{k-1} \setminus U_t$. Then $v_t \in \mathscr{C}(U_0 \cap \ldots \cap U_{k-1})$ since $\varphi_t$ has compact support in $U_t$, and $h_{U_0,\dots,U_{k-1}} = \sum_{t\in T} v_t$ is continuous on $U_0 \cap \ldots \cap U_{k-1}$ since $(\varphi_t)_{t\in T}$ is locally finite. With $V_{0,t} = U_t$ and $V_{k,t} = U_{k+1}$ we then have on $U_0 \cap \ldots \cap U_{k-1}$

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^j h_{U_0,\dots,\widehat{U}_j,\dots U_k} = \sum_{t\in T}\sum_{j=0}^{k}(-1)^j \varphi_t f_{U_t,U_0,\dots,\widehat{U}_j,\dots,U_k} \\
&= \sum_{t\in T}\varphi_t \sum_{j=0}^{k+1}(-1)^{j+1} f_{V_{0,t},\dots,\widehat{V}_{j,t},\dots,V_{k+1,t}} + \sum_{t\in T}\varphi_t f_{U_0,\dots,U_k} \\
&= f_{U_0,\dots,U_k}.
\end{aligned}$$

This proves $\ker(d_k) = \operatorname{im}(d_{k-1})$.

Of course, we can replace $\mathscr{C}$ in the example by any sheaf which permits partitions of unity, e.g. the sheaf of $\mathscr{C}^\infty$-functions if $X$ is a $\mathscr{C}^\infty$- manifold.

**Definition 4.1.6** *Let $\mathscr{X} = (X_\alpha, \varrho_\beta^\alpha)$ and $\mathscr{Y} = (Y_s, \sigma_t^s)$ be projective $I$- (respectively $T$-) spectra.*

*A map $F = (\varphi, f) : \mathscr{X} \longrightarrow \mathscr{Y}$ consists of an increasing function $\varphi : T \longrightarrow I$ and a family $f = (f_t)_{t\in T}$ of linear maps $f_t : X_{\varphi(t)} \longrightarrow Y_t$ such that $f_s \circ \varrho_{\varphi(t)}^{\varphi(s)} = \sigma_t^s \circ f_t$ for all $s \leq t$. The composition $G \circ F$ of two maps $F$ and $G$ is defined in the obvious way, and it is clear that this operation is associative and has an identity.*

*Two maps $F = (\varphi, f)$, $G : (\gamma, g) : \mathscr{X} \longrightarrow \mathscr{Y}$ are equivalent (we then write $F \sim G$) if there is an increasing function $\delta : T \longrightarrow I$ with $\delta \geq \varphi$, $\delta \geq \gamma$ and $f_s \circ \varrho_{\delta(s)}^{\varphi(s)} = g_s \circ \varrho_{\delta(s)}^{\gamma(s)}$ for all $s \in T$.*

*$\mathscr{X}$ and $\mathscr{Y}$ are equivalent if there are maps $F : \mathscr{X} \longrightarrow \mathscr{Y}$ and $G : \mathscr{Y} \longrightarrow \mathscr{X}$ such that $G \circ F$ and $F \circ G$ are equivalent to the identity maps on $\mathscr{X}$ and $\mathscr{Y}$ respectively.*

A map $F = (\varphi, f) : \mathscr{X} \longrightarrow \mathscr{Y}$ induces a cochain complex map between the complexes defined in 4.1.3, i.e. there is a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \prod\limits_{\alpha_0} X_{\alpha_0} & \xrightarrow{d_0} & \prod\limits_{\alpha_0 \leq \alpha_1} X_{\alpha_0} & \xrightarrow{d_1} & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \prod\limits_{s_0} Y_{s_0} & \longrightarrow & \prod\limits_{s_0 \leq s_1} Y_{s_0} & \longrightarrow & \cdots
\end{array}$$

where the vertical maps $\prod\limits_{\alpha_0 \leq \ldots \leq \alpha_k} X_{\alpha_0} \longrightarrow \prod\limits_{s_0 \leq \ldots \leq s_k} Y_{s_0}$ are given by

$$(x_{\alpha_0,\ldots,\alpha_k})_{\alpha_0 \leq \ldots \leq \alpha_k} \longmapsto \left(f_{s_0}(x_{\varphi(s_0),\ldots,\varphi(s_k)})\right)_{s_0 \leq \ldots \leq s_k}.$$

Hence $F$ induces natural maps $f^k : \operatorname{Proj}^k \mathscr{X} \longrightarrow \operatorname{Proj}^k \mathscr{Y}$ between the cohomology groups. $f^k$ is explicitely defined by

$$(x_{\alpha_0,\ldots,\alpha_k})_{\alpha_0 \leq \ldots \alpha_k} + \operatorname{im}(d_{k-1}) \mapsto \left(f_{s_0}(x_{\varphi(s_0),\ldots \varphi(s_k)})\right)_{s_0 \leq \ldots \leq s_k} + \operatorname{im}(d_k),$$

and for maps $F = (\varphi, f) : \mathscr{X} \longrightarrow \mathscr{Y}$ and $G : (\gamma, g) : \mathscr{Y} \longrightarrow \mathscr{Z}$ we have

$$(g \circ f)^k = g^k \circ f^k : \operatorname{Proj}^k \mathscr{X} \longrightarrow \operatorname{Proj}^k \mathscr{Z}.$$

**Theorem 4.1.7** *If $\mathscr{X}$ and $\mathscr{Y}$ are equivalent projective spectra then we have $\operatorname{Proj}^k \mathscr{X} \cong \operatorname{Proj}^k \mathscr{Y}$ for all $k \in \mathbb{N}_0$.*

The proof requires some calculations which are contained in the next lemma. Again, we use the notation of theorem 4.1.3.

**Lemma 4.1.8** *Let $\varphi : I \longrightarrow I$ be an increasing map with $\varphi(\alpha) \geq \alpha$ for all $\alpha \in I$ and $x = (x_{\alpha_0,\ldots,\alpha_k})_{\alpha_0 \leq \ldots \leq \alpha_k} \in \ker(d_k)$. Then*

$$\left(\varrho_{\varphi(\alpha_0)}^{\alpha_0}(x_{\varphi(\alpha_0),\ldots,\varphi(\alpha_k)}) - x_{\alpha_0,\ldots,\alpha_k}\right)_{\alpha_0 \leq \ldots \leq \alpha_k} \in \operatorname{im}(d_{k-1}).$$

**Proof.** To simplify the expressions we will again omit the spectral maps. For $\alpha_0 \leq \ldots \leq \alpha_{k-1}$ we define

$$y_{\alpha_0,\ldots,\alpha_{k-1}} = \sum_{l=0}^{k-1} (-1)^l x_{\alpha_0,\ldots,\alpha_l,\varphi(\alpha_l),\ldots,\varphi(\alpha_{k-1})}.$$

For fixed $\alpha_0 \leq \ldots \leq \alpha_k$ and $0 \leq l \leq k$ we set

$$\beta^l = (\alpha_0, \ldots, \alpha_l, \varphi(\alpha_l), \ldots, \varphi(\alpha_k)), \text{ i.e.}$$

$$\beta^l_j = \begin{cases} \alpha_j & , 0 \leq j \leq l \\ \varphi(\alpha_{j-1}) & , l < j \leq k+1 \end{cases}$$

We then have $(*)$ $(\beta^l_0, \ldots, \widehat{\beta}^l_{l+1}, \ldots, \beta^l_{k+1}) = (\beta^{l+1}_0, \ldots, \widehat{\beta}^{l+1}_{l+1}, \ldots, \beta^{l+1}_{k+1})$ for $0 \leq l \leq k-1$. Since $d_k(x) = 0$ we have

$$(-1)^l \sum_{j=0}^{k+1} (-1)^j x_{\beta^l_0,\ldots,\widehat{\beta}^l_j,\ldots,\beta^l_{k+1}} = 0$$

for all $l \leq k$. If we sum with respect to $l$ we see with $(*)$ that the terms with $j = l$ and $j = l+1$ cancel out expect for $j = l = 0$ and $j = l+1 = k+1$ for which we get $x_{\varphi(\alpha_0),\ldots,\varphi(\alpha_k)}$ and $-x_{\alpha_0,\ldots,\alpha_k}$. We thus obtain

$$\begin{aligned}
&-(x_{\varphi(\alpha_0),\ldots,\varphi(\alpha_k)} - x_{\alpha_0,\ldots,\alpha_k}) \\
&= \sum_{j=0}^{k+1} \left( \sum_{l=0}^{j-2} + \sum_{l=j+1}^{k} \right) (-1)^{l+j} x_{\beta^l_0,\ldots,\widehat{\beta}^l_j,\ldots,\beta^l_{k+1}} \\
&= \sum_{i=0}^{k} \sum_{l=0}^{i-1} (-1)^{l+i+1} x_{\beta^l_0,\ldots,\widehat{\beta}^l_{i+1},\ldots,\beta^l_{k+1}} + \sum_{j=0}^{k} \sum_{l=j+1}^{k} (-1)^{l+j} x_{\beta^l_0,\ldots,\widehat{\beta}^l_j,\ldots,\beta^l_{k+1}} \\
&= -\sum_{i=0}^{k} (-1)^i \left( \sum_{l=0}^{i-1} (-1)^l x_{\beta^l_0,\ldots,\widehat{\beta}^l_{i+1},\ldots,\beta^l_{k+1}} - \sum_{l=i+1}^{k} (-1)^l x_{\beta^l_0,\ldots,\widehat{\beta}^l_i,\ldots,\beta^l_{k+1}} \right) \\
&= -\sum_{i=0}^{k} (-1)^i \left( \sum_{l=0}^{i-1} (-1)^l x_{\alpha_0,\ldots,\alpha_l,\varphi(\alpha_l),\ldots,\widehat{\varphi(\alpha_i)},\ldots,\varphi(\alpha_k)} \right. \\
&\qquad \left. - \sum_{l=i+1}^{k} (-1)^l x_{\alpha_0,\ldots,\widehat{\alpha}_i,\ldots,\alpha_l,\varphi(\alpha_l),\ldots\varphi(\alpha_k)} \right) \\
&= -\sum_{i=0}^{k} (-1)^i y_{\alpha_0,\ldots,\widehat{\alpha}_i,\ldots,\alpha_k}.
\end{aligned}$$

□

**Proof of theorem 4.1.7.** We will show that for a map $F = (\varphi, f) : \mathscr{X} \longrightarrow \mathscr{X}$ which is equivalent to the identity map $\mathscr{X} \longrightarrow \mathscr{X}$ we have $f^k = id : \operatorname{Proj}^k \mathscr{X} \longrightarrow \operatorname{Proj}^k \mathscr{X}$ where $f^k$ is defined as above. This means that for each $x = (x_{\alpha_0,\ldots,\alpha_k})_{\alpha_0 \leq \ldots \leq \alpha_k} \in \ker(d_k)$ we have to find $y = (y_{\alpha_0,\ldots,\alpha_{k-1}})_{\alpha_0 \leq \ldots \leq \alpha_{k-1}}$ with $d_{k-1}(y) = f_{\alpha_0}\left(x_{\varphi(\alpha_0),\ldots,\varphi(\alpha_k)}\right)_{\alpha_0 \leq \ldots \leq \alpha_k} - x$. There is an increasing map $\psi : I \longrightarrow I$ with $\psi \geq \varphi$, $\psi(\alpha) \geq \alpha$, and $f_\alpha \circ \varrho^{\varphi(\alpha)}_{\psi(\alpha)} = \varrho^{\alpha}_{\psi(\alpha)}$. We apply 4.1.8 to find $(u_{\varphi(\alpha_0),\ldots,\varphi(\alpha_{k-1})})_{\alpha_0 \leq \ldots \leq \alpha_{k-1}}$ with

$$x_{\varphi(\alpha_0),\ldots,\varphi(\alpha_k)} - \varrho^{\varphi(\alpha_0)}_{\psi(\alpha_0)}\left(x_{\psi(\alpha_0),\ldots,\psi(\alpha_k)}\right) = \sum_{j=0}^{k} (-1)^j u_{\varphi(\alpha_0),\ldots,\widehat{\varphi(\alpha_j)},\ldots,\varphi(\alpha_k)}$$

and to find $(v_{\alpha_0,\ldots,\alpha_{k-1}})_{\alpha_0 \leq \ldots \leq \alpha_{k-1}}$ with

$$\varrho^{\alpha_0}_{\psi(\alpha_0)}\left(x_{\psi(\alpha_0),\ldots,\psi(\alpha_k)}\right) - x_{\alpha_0,\ldots,\alpha_k} = \sum_{j=0}^{k} (-1)^j v_{\alpha_0,\ldots,\widehat{\alpha}_j,\ldots,\alpha_k}.$$

Then $y = \left(v_{\alpha_0,\ldots,\alpha_{k-1}} + f_{\alpha_0} u_{\varphi(\alpha_0),\ldots,\varphi(\alpha_{k-1})}\right)_{\alpha_0 \leq \ldots \leq \alpha_{k-1}}$ satisfies

$$d_{k-1}(y) = \left(f_{\alpha_0} x_{\varphi(\alpha_0),\ldots,\varphi(\alpha_k)} - x_{\alpha_0,\ldots,\alpha_k}\right)_{\alpha_0 \leq \ldots \leq \alpha_k}.$$

□

We now present a simple direct proof of a result due to B. Mitchell [47].

**Theorem 4.1.9** *Let $\mathscr{X} = (x_\alpha, \varrho^\alpha_\beta)$ be a projective $I$-spectrum with $|I| \leq \aleph_n$, the $n$-th infinite cardinal number. Then* $\operatorname{Proj}^k \mathscr{X} = 0$ *for all $k \geq n+2$.*

**Proof.** We prove the result by induction on $n \in \mathbb{N}_0$. For $n = 0$ it is easy to construct an $\mathbb{N}$-spectrum $\mathscr{Y} \sim \mathscr{X}$. Then 4.1.7 yields $\operatorname{Proj}^k \mathscr{X} \cong \operatorname{Proj}^k \mathscr{Y} = 0$ for $k \geq 2$ by theorem 3.1.4. We now suppose that the result is true for $J$-spectra with $|J| \leq \aleph_{n-1}$. If $|I| = \aleph_n$ we can find a limit ordinal $\omega$ and $I_\nu \subset I_\omega := I$ for $\nu < \omega$ such that $|I_\nu| < |I|$, $I_\nu \subseteq I_\mu$ for $\nu < \mu$ and $I_\mu = \bigcup_{\nu<\mu} I_\nu$ for limit ordinals $\mu \leq \omega$.

Since $I$ is directed there is $f : I \times I \longrightarrow I$ with $f(\alpha,\beta) \geq \alpha$ and $f(\alpha,\beta) \geq \beta$ for all $\alpha, \beta \in I$. If $M$ is any subset of $I$ we set $M^0 = M$, $M^{l+1} = \{f(\alpha,\beta) : \alpha, \beta \in M^l\}$, and $\tilde{M} = \bigcup_{l \in \mathbb{N}_0} M_l$ which is at most countable if $M$ is finite, and else has the same cardinality as $M$. What we have gained by this construction is that $(\tilde{M}, \leq)$ is a directed subset of $I$.

Now, the family $(J_\nu = \tilde{I}_\nu)_{\nu \leq \omega}$, has the same properties as $(I_\nu)_{\nu \leq \omega}$. To show $\operatorname{Proj}^k \mathscr{X} = 0$ for $k \geq n+2$ we use again theorem 4.1.3. Let $x = (x_{\alpha_0,\ldots,\alpha_k})_{\alpha_0 \leq \ldots \leq \alpha_k} \in \ker(d_k)$ be given. We construct by transfinite induction $y^\nu = (y^\nu_{\alpha_0,\ldots,\alpha_{k-1}})_{\alpha_0 \leq \ldots \leq \alpha_{k-1} \in J_\nu}$ with

1. $x_{\alpha_0,\ldots,\alpha_k} = \sum_{j=0}^{k} (-1)^j y^\nu_{\alpha_0,\ldots,\widehat{\alpha}_j,\ldots,\alpha_k}$ for $\alpha_0 \leq \ldots \leq \alpha_k \in J_\nu$ and

2. $y^{\nu}_{\alpha_0,\dots,\alpha_{k-1}} = y^{\lambda}_{\alpha_0,\dots,\alpha_{k-1}}$ for $\lambda < \nu$ and $\alpha_0,\dots,\alpha_{k-1} \in J_\lambda$.

If $\nu$ is a limit ordinal there is nothing to do since $J_\nu = \bigcup_{\lambda<\nu} J_\lambda$. Let now $\nu = \lambda+1$ be a successor ordinal and denote by $\mathscr{X}^\nu$ the $J_\nu$-spectrum $(X_\alpha, \varrho^\alpha_\beta)_{\alpha\in J_\nu}$. By induction hypothesis we have $\operatorname{Proj}^k \mathscr{X}^\nu = 0$ hence there is

$$z = (z_{\alpha_0,\dots,\alpha_{k-1}})_{\alpha_0\le\dots\le\alpha_{k-1}\in J_\nu}$$

with $x_{\alpha_0,\dots,\alpha_k} = \sum_{j=0}^{k} (-1)^j z_{\alpha_0,\dots,\widehat{\alpha}_j,\dots,\alpha_k}$ for $\alpha_0,\dots,\alpha_k \in J_\nu$. Now

$$u_{\alpha_0,\dots,\alpha_{k-1}} = y^{\lambda}_{\alpha_0,\dots,\alpha_{k-1}} - z_{\alpha_0,\dots,\alpha_{k-1}} \text{ for } \alpha_0 \le \dots \le \alpha_{k-1} \in J_\lambda$$

defines an element $u \in \ker(d^\lambda_{k-1})$ (where $d^\lambda_k$ are the differentials defined in 4.1.3 corresponding to $\mathscr{X}^\lambda$). Since $|J_\lambda| \le \aleph_{n-1}$ we have $\operatorname{Proj}^{k-1}\mathscr{X}^\lambda = 0$, hence there is $v = (v_{\alpha_0,\dots,\alpha_{k-2}})_{\alpha_0\le\dots\le\alpha_{k-2}\in J_\lambda}$ with $d^\lambda_{k-2}(v) = u$. We now define $\tilde{v}$ by

$$\tilde{v}_{\alpha_0,\dots,\alpha_{k-2}} = \begin{cases} v_{\alpha_0,\dots,\alpha_{k-2}} & \text{if } \alpha_0,\dots\alpha_{k-2} \in J_\lambda \\ 0 & \text{else} \end{cases}$$

and $y^\nu = z + d^\nu_{k-2}(\tilde{v})$. Then $y^\nu$ satisfies 1. and 2. Finally, $y_{\alpha_0,\dots,\alpha_{k-1}} := y^\nu_{\alpha_0,\dots,\alpha_{k-1}}$ for $\alpha_0 \le \dots \le \alpha_{k-1} \in J_\nu$ defines a solution $y$ of $d_{k-1}(y) = x$. □

Using the technique of the preceding proof we also obtain the following result of Jensen [37, théorème 1.8].

**Theorem 4.1.10** *Let $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ be a projective $I$-spectrum such that for each directed subset $J \subset I$ the restriction* $\operatorname{Proj}\mathscr{X} \longrightarrow \operatorname{Proj}\mathscr{X}^J$ *is surjective (where $\mathscr{X}^J$ is the $J$-spectrum $(X_\alpha, \varrho^\alpha_\beta)_{\alpha\in J}$). Then* $\operatorname{Proj}^k\mathscr{X} = 0$ *for all $k \ge 1$.*

**Proof.** We proceed by induction on $k \in \mathbb{N}$. For $k = 1$ the theorem is proved by induction on the cardinality of $I$. Let us thus assume $\operatorname{Proj}^1\mathscr{Y} = 0$ for all $J$-spectra $\mathscr{Y}$ satisfying the assumption of the theorem where the cardinality of $J$ is strictly smaller than that of $I$. Let now $(J_\nu)_{\nu<\omega}$ be as in the previous proof and $x = (x_{\alpha_0,\alpha_1})_{\alpha_0\le\alpha_1} \in \ker d_1$. We construct by transfinite induction $(y^\nu_{\alpha_0})_{\alpha_0\in J_\nu}$ with

1. $x_{\alpha_0,\alpha_1} = \varrho^{\alpha_0}_{\alpha_1}(y^\nu_{\alpha_1}) - y^\nu_{\alpha_0}$ for $\alpha_0 \le \alpha_1 \in J_\nu$ and
2. $y^\nu_{\alpha_0} = y^\lambda_{\alpha_0}$ for $\lambda < \nu$ and $\alpha_0 \in J_\lambda$.

For limit ordinals $\nu$ there is nothing to do since $J_\nu = \bigcup_{\lambda<\nu} J_\lambda$. Let now $\nu = \lambda + 1$ be a successor ordinal. Since the cardinality of $J_\nu$ is strictly smaller than that of $I$ we have $\operatorname{Proj}^1\mathscr{X}^{J_\nu} = 0$, hence there are $(z_{\alpha_0})_{\alpha_0\in J_\nu}$ with $x_{\alpha_0,\alpha-1} = \varrho^{\alpha_0}_{\alpha_1}(z_{\alpha_1}) - z_{\alpha_0}$ for $\alpha_0 \le \alpha_1 \in J_\nu$. Defining $u_{\alpha_0} = y^\lambda_{\alpha_0} - z_{\alpha_0}$ for $\alpha_0 \in J_\lambda$ we obtain $(u_{\alpha_0})_{\alpha_0\in J_\lambda} \in \operatorname{Proj}\mathscr{X}^{J_\lambda}$ and since the restriction $\operatorname{Proj}\mathscr{X} \longrightarrow \operatorname{Proj}\mathscr{X}^{J_\lambda}$ is surjective there is $(v_{\alpha_0})_{\alpha_0\in I}$ with $v_{\alpha_0} = u_{\alpha_0}$ for $\alpha_0 \in J_\lambda$. Thus, $y^\nu_{\alpha_0} := z_0 + v_{\alpha_0}, \alpha_0 \in J_\nu$ satisfies 1. and 2.

The induction step $k \longrightarrow k+1$ is exactly as in 4.1.9. □

## 4.2 Insertion: The completion functor

For a locally convex space $X$ let $\widetilde{X}$ be the Hausdorff completion of $X$, i.e. the completion of $X/\overline{\{0\}}$ and $j_X : X \longrightarrow \widetilde{X}$ the canonical map (which is injective iff $X$ is Hausdorff and surjective iff $X$ is complete). Let $\mathscr{C}$ be the functor assigning to $X$ the space $\widetilde{X}$ (ignoring its topology) and to a continuous linear map $f : X \longrightarrow Y$ the canonical map $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$. Then it is easily seen that $\mathscr{C}$ is injective. The derived functors of $\mathscr{C}$ are denoted by $\mathscr{C}^k$.

We recall that any vector space endowed with the coarsest topology is injective. Applying $\mathscr{C}$ to the exact sequence

$$0 \longrightarrow \overline{\{0\}}^X \longrightarrow X \longrightarrow X/\overline{\{0\}} \longrightarrow 0$$

yields $\mathscr{C}^k(X) = \mathscr{C}^k(X/\overline{\{0\}})$ for all $k \in \mathbb{N}_0$, and applying $\mathscr{C}$ to the exact sequence

$$0 \longrightarrow X/\overline{\{0\}} \longrightarrow \widetilde{X} \longrightarrow Z \longrightarrow 0$$

(with $Z = \widetilde{X}/\left(X/\overline{\{0\}}\right)$ carrying the coarsest topology as $X/\overline{\{0\}}$ is dense in $\widetilde{X}$) yields $\mathscr{C}^k(\widetilde{X}) = \mathscr{C}^k(X/\overline{\{0\}}) = \mathscr{C}^k(X)$ for all $k \in \mathbb{N}_0$. We have the following result of Palamodov [50, propositions 10.2 and 10.4].

**Theorem 4.2.1** *If $X$ is a semi-metrizable locally convex space then $\mathscr{C}^k(X) = 0$ holds for all $k \geq 1$.*

**Proof.** By what we have said above, we have to show this for $Y = \widetilde{X}$ which is a Fréchet space and thus, $Y$ has an injective resolution

$$0 \longrightarrow Y \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow \ldots$$

consisting of Fréchet spaces. Since $Y$ and $Y_j$ are complete and Hausdorff, the application of $\mathscr{C}$ does not change this complex (except ignoring the topologies). This proves the theorem. □

An immediate consequence is the following result.

**Corollary 4.2.2** *If a locally convex space is isomorphic to a product of semi-metrizable locally convex spaces then $\mathscr{C}^k(X) = 0$ for all $k \geq 1$.*

In the list of unsolved problems Palamodov asked whether $\mathscr{C}^1(\mathscr{D}) = 0$ where $\mathscr{D}$ is the (LF)-space of $\mathscr{C}^\infty$-functions with compact support, and whether $\mathscr{C}^1(\varphi) = 0$ where $\varphi$ is the space of finite sequences. We provide the answers to these questions at least under some set-theoretic assumptions in the next section. Let us mention that the problem when the quotient of complete locally convex spaces or even topological groups is again complete has been investigated several times in the literature (in most cases without using the language of derived functors). In particular, it is a classical result that the quotient of a complete topological group modulo a normal metrizable subgroup is complete. (If the group itself is metrizable this follows from the Schauder lemma:

the composition of the quotient map with the embedding of the quotient into its completion is almost open and therefore open and surjective. The general case can be deduced from this.)

Moreover, D. Wigner [72] considered the completion functor in the category of abelian topological groups and obtained the analogous results of 4.2.1 and 4.2.2. Since the category of abelian topological groups does not have many injective objects the derived functors $\mathscr{C}^k$ cannot be defined as in our case. Wigner used so-called Yoneda groups for the definition of the derived functors.

## 4.3 Projective spectra of locally convex spaces

We now consider locally convex projective $I$-spectra $\mathscr{X} = (X_\alpha, \varrho_\beta^\alpha)$, i.e. $\mathscr{X}$ is a projective $I$-spectrum consisting of locally convex spaces $X_\alpha$ and continuous linear maps $\varrho_\beta^\alpha$. A morphism in this category is an "algebraic morphism" with continuous components. The projective limit is endowed with the relative topology of the product. A basis of $\mathscr{U}_0(\operatorname{Proj}\mathscr{X})$ is given by $\{(\varrho^\alpha)^{-1}(U) : \alpha \in I, U \in \mathscr{U}_0(X_\alpha)\}$. As in section 3.3 we have:

**Theorem 4.3.1** *For a locally convex $I$-spectrum $\mathscr{X} = (X_\alpha, \varrho_\beta^\alpha)$ the following conditions are equivalent.*

1. $d_0 : \prod_{\alpha\in I} X_\alpha \longrightarrow \prod_{\alpha_0\le\alpha_1} X_{\alpha_0},\ (x_\alpha)_{\alpha\in I} \mapsto (\varrho_{\alpha_1}^{\alpha_0}(x_{\alpha_1}) - x_{\alpha_0})_{\alpha_0\le\alpha_1}$ *is open onto its range.*
2. $\forall\, \alpha \in I, U \in \mathscr{U}_0(X_\alpha)\ \exists\, \beta \ge \alpha \quad \varrho_\beta^\alpha(X_\beta) \subseteq \varrho^\alpha(\operatorname{Proj}\mathscr{X}) + U$.
3. *For every exact sequence*

$$0 \longrightarrow \mathscr{X} \longrightarrow \mathscr{Y} \longrightarrow \mathscr{Z} \longrightarrow 0$$

*of locally convex $I$-spectra the induced map* $\operatorname{Proj}\mathscr{Y} \longrightarrow \operatorname{Proj}\mathscr{Z}$ *is open onto its range.*

**Proof.** If 1. holds and $U \in \mathscr{U}_0(X_\alpha)$ for some $\alpha \in I$ is given there is a finite set $J \subseteq \{(\alpha_0, \alpha_1) \in I^2 : \alpha_0 \le \alpha_1\} =: I_+^2$ such that

$$\left(\prod_{(\alpha_0,\alpha_1)\in J} \{0\} \times \prod_{(\alpha_0,\alpha_1)\in I_+^2\setminus J} X_{\alpha_0}\right) \cap \operatorname{Im}(d_1) \subseteq d_1\left(U \times \prod_{\alpha_0\in I\setminus\{\alpha\}} X_{\alpha_0}\right).$$

We choose $\beta \in I$ with $\beta \ge \alpha$ and $\beta \ge \alpha_1$ for all $(\alpha_0, \alpha_1) \in J$. Given $x_\beta \in \mathscr{X}_\beta$ we set

$$x_\gamma := \begin{cases} \varrho_\beta^\gamma x_\beta \text{ if } \gamma \le \beta \\ \ \ 0 \quad \text{else} \end{cases} \quad \text{and } x = (x_\gamma)_{\gamma\in I}.$$

Then there is $y = (y_{\alpha_0})_{\alpha_0\in I} \in \prod_{\alpha_0\in I} X_{\alpha_0}$ with $y_\alpha \in U$ such that $d_0(x) = d_0(y)$. This yields $\varrho_\beta^\alpha x_\beta - y_\alpha \in \varrho^\alpha(\operatorname{Proj}\mathscr{X})$. Let now 2. be satisfied and

$$0 \longrightarrow \mathscr{X} \xrightarrow{\ f\ } \mathscr{Y} \xrightarrow{\ g\ } \mathscr{Z} \longrightarrow 0$$

be an exact sequence of locally convex $I$-spectra with $\mathscr{Y} = (Y_\alpha, \sigma_\beta^\alpha)$ and $\mathscr{Z} = (Z_\alpha, \tau_\beta^\alpha)$. We may assume that $X_\alpha$ is a topological subspace of $Z_\alpha$, $\varrho_\beta^\alpha = \sigma_\beta^\alpha|_{X_\beta}$, and that $g_\alpha$ is the quotient map $Z_\alpha \longrightarrow Z_\alpha/X_\alpha$. We denote $\operatorname{Proj}(g)$ by $g^* : \operatorname{Proj}\mathscr{Y} \longrightarrow \operatorname{Proj}\mathscr{Z}$. Given $U \in \mathscr{U}_0(\operatorname{Proj}\mathscr{Z})$ there are $\alpha \in I$ and $V \in \mathscr{U}_0(Y_\alpha)$ with $(\sigma^\alpha)^{-1}(V + V) \subseteq U$. We choose $\beta \ge \alpha$ such that $\varrho_\beta^\alpha(X_\beta) \subseteq \varrho^\alpha(\operatorname{Proj}\mathscr{X}) + V$ and $W \in \mathscr{U}_0(Y_\beta)$ with $\sigma_\beta^\alpha(W) \subseteq V$.

We show that $g^*(U)$ contains $M := (\tau^\beta)^{-1}(g_\beta(W)) \cap \operatorname{Im}(g^*)$ which is a 0-neighbourhood of $\operatorname{Im}(g^*)$ with the topology induced by $\operatorname{Proj}\mathscr{Z}$ since $g_\beta$ is open. Given $z = (z_\gamma)_{\gamma\in I} \in M$ there are $y = (y_\gamma)_{\gamma\in I} \in \operatorname{Proj}\mathscr{Y}$ and $w_\beta \in W$ with $g_\gamma(y_0) = z_\gamma$ for all $\gamma \in I$ and $g_\beta(w_\beta) = y_\gamma$. Hence, $y_\beta - w_\beta \in \ker(g_\beta) = X_\beta$ and we can find $v_\alpha \in V$ and $x = (x_\gamma)_{\gamma\in I} \in \operatorname{Proj}\mathscr{X}$ with $\sigma^\alpha_\beta(y_\beta - w_\beta) = v_\alpha + x_\alpha$. We conclude $z = g^*(y-x)$ and $y - x \in U$ since $y_\alpha - x_\alpha = \sigma^\alpha_\beta(w_\beta) + v_\alpha \in V + V$.

3. implies 1. by considering the particular exact sequence

$$0 \longrightarrow \mathscr{X} \stackrel{i}{\longrightarrow} \mathscr{F}_0 \longrightarrow \operatorname{coker} i \longrightarrow 0$$

where $\mathscr{F}_0$ and $i$ are as in the constructions preceding 4.1.3. □

Let us say $\operatorname{Proj}^+\mathscr{X} = 0$ if the conditions of 4.3.1 hold. As in section 3.3 this is equivalent to $\operatorname{Pr}^+_M\mathscr{X} = 0$ for all sets $M$, where $\operatorname{Pr}_M = \operatorname{Proj}\circ H_M$. It is easily seen (e.g. via condition 2. or the general result 2.2.2) that $\operatorname{Proj}^+\mathscr{X} = 0$ implies $\operatorname{Proj}^+\mathscr{Y} = 0$ for every $I$-spectrum $\mathscr{Y}$ which is a quotient of $\mathscr{X}$ (i.e. there is an epihomomorphism from $\mathscr{X}$ to $\mathscr{Y}$). Let

$$0 \longrightarrow \mathscr{X} \stackrel{i}{\longrightarrow} \mathscr{F}_0 \stackrel{i_0}{\longrightarrow} \mathscr{F}_1 \stackrel{i_1}{\longrightarrow} \mathscr{F}_2 \longrightarrow \ldots$$

be the injective resolution constructed in 4.1. We then have $\operatorname{Proj}^+\mathscr{F}_k = 0$ for all $k \in \mathbb{N}_0$ since condition 2. is trivially satisfied for all $k \in \mathbb{N}_0$, hence $\operatorname{Proj}^+(\ker i_k) = \operatorname{Proj}^+(\operatorname{coker} i_{k-1}) = 0$ for all $k \geq 1$. Applying 4.3.1 to the exact sequence

$$0 \longrightarrow \ker i_k \longrightarrow \mathscr{F}_k \stackrel{i_k}{\longrightarrow} \ker(i_{k+1}) \longrightarrow 0$$

we obtain that $d_k = \operatorname{Proj}(i_k) : \prod_{\alpha_0\leq\ldots\leq\alpha_k} X_{\alpha_0} \longrightarrow \prod_{\alpha_0\leq\ldots\leq\alpha_{k-1}} X_{\alpha_0}$ is always open onto its range for $k \geq 1$.

The next result is a variant of a theorem due to Wigner [72].

**Proposition 4.3.2** *Let $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ be a locally convex projective $I$-spectrum with $\operatorname{Proj}^+\mathscr{X} = 0$. If all spaces $X_\alpha$ are complete and satisfy $\mathscr{C}^k(X_\alpha) = 0$ for all $k \geq 1$ then we have $\operatorname{Proj}^k\mathscr{X} = \mathscr{C}^k(\operatorname{Proj}\mathscr{X})$ for all $k \geq 1$.*

**Proof.** The proof is by induction on $k$. We consider the exact sequence

$$0 \longrightarrow \mathscr{X} \stackrel{i}{\longrightarrow} \mathscr{F}_0 \stackrel{i_0}{\longrightarrow} \mathscr{Z} \longrightarrow 0$$

where as above, $\mathscr{F}_0$ is the spectrum consisting of the products $\prod_{\alpha_0\leq\alpha} X_{\alpha_0}$ and the canonical projections and $\mathscr{Z}$ is the cokernel spectrum.

It is easily seen that $d_0 = \operatorname{Proj} i_0 : \operatorname{Proj}\mathscr{F}_0 \longrightarrow \operatorname{Proj}\mathscr{Z}$ has dense range and by 4.3.1 $d_0$ is open onto its range, hence $\operatorname{Proj}\mathscr{Z} = \widetilde{Q}$ where $Q = \prod_{\alpha\in I} X_\alpha/\operatorname{Proj}\mathscr{X}$. Applying $\mathscr{C}$ to the exact sequence

$$0 \longrightarrow \operatorname{Proj}\mathscr{X} \longrightarrow \prod_{\alpha\in I} X_\alpha \longrightarrow Q \longrightarrow 0$$

we obtain the exact sequence

$$0 \longrightarrow \operatorname{Proj} \mathscr{X} \longrightarrow \prod_{\alpha \in I} X_\alpha \longrightarrow \operatorname{Proj} \mathscr{Z} \longrightarrow \mathscr{C}^1(\operatorname{Proj} \mathscr{X}) \longrightarrow 0$$

since $\mathscr{C}^1(\prod_{\alpha \in I} X_\alpha) = 0$. On the other hand we apply Proj to

$$0 \longrightarrow \mathscr{X} \longrightarrow \mathscr{F}_0 \longrightarrow \mathscr{Z} \longrightarrow 0$$

and obtain the exact sequence

$$0 \longrightarrow \operatorname{Proj} \mathscr{X} \longrightarrow \prod_{\alpha \in I} X_\alpha \longrightarrow \operatorname{Proj} \mathscr{Z} \longrightarrow \operatorname{Proj}^1 \mathscr{X} \longrightarrow 0.$$

This gives $\mathscr{C}^1(\operatorname{Proj} \mathscr{X}) \cong \operatorname{Proj}^1 \mathscr{X}$.

For the induction step we first note that $\operatorname{Proj}^{k+1} \mathscr{X} = \operatorname{Proj}^k \mathscr{Z}$. Moreover, $\mathscr{Z}$ consists of $Z_\alpha = \prod_{\alpha_0 \leq \alpha} X_{\alpha_0}/X_\alpha$ and these spaces are complete (since $\mathscr{C}^1(X_\alpha) = 0$) and satisfy $\mathscr{C}^k(Z_\alpha) = \mathscr{C}^{k+1}(X_\alpha) = 0$ for all $k \in \mathbb{N}$.

As we have noted above, $\operatorname{Proj}^+ \mathscr{Z} = 0$, hence $\mathscr{Z}$ satisfies the same assumption as $\mathscr{X}$ and we conclude from the induction hypothesis $\operatorname{Proj}^{k+1} \mathscr{X} = \operatorname{Proj}^k(\mathscr{Z}) = \mathscr{C}^k(\operatorname{Proj} \mathscr{Z}) = \mathscr{C}^k(\widetilde{Q}) = \mathscr{C}^k(Q) = \mathscr{C}^{k+1}(\operatorname{Proj} \mathscr{X})$. □

The following corollaries are immediate consequences. Note however that they cannot be proved itself by the induction procedure above since $Z_\alpha = \prod_{\alpha_0 \leq \alpha} X_{\alpha_0}/X_\alpha$ does not inherit all good properties form $X_\alpha$.

**Corollary 4.3.3** *Let $\mathscr{X}$ be a locally convex projective $I$-spectrum consisting of Fréchet spaces such that* $\operatorname{Proj}^+ \mathscr{X} = 0$. *Then* $\operatorname{Proj}^k \mathscr{X} = \mathscr{C}^k(\operatorname{Proj}\mathscr{X})$ *for all* $k \geq 1$.

**Corollary 4.3.4** *Let $\mathscr{X}$ and $\mathscr{Y}$ be projective $I$-(respectively $J$-) spectra both consisting of Fréchet spaces such that* $\operatorname{Proj}^+ \mathscr{X} = 0$ *and* $\operatorname{Proj}^+ \mathscr{Y} = 0$. *If the locally convex spaces* $\operatorname{Proj} \mathscr{X}$ *and* $\operatorname{Proj} \mathscr{Y}$ *are isomorphic then* $\operatorname{Proj}^k \mathscr{X} \cong \operatorname{Proj}^k \mathscr{Y}$ *for all* $k \geq 1$.

As in the countable case, we call a locally convex projective $I$-spectrum $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ reduced if

$$\forall\, \alpha \in I \;\exists\, \beta \geq \alpha \;\; \forall\, \gamma \geq \beta \quad \varrho^\alpha_\beta(X_\beta) \subseteq \overline{\varrho^\alpha_\gamma(X_\gamma)}$$

and strongly reduced if

$$\forall\, \alpha \in I \;\exists\, \beta \geq \alpha \quad \varrho^\alpha_\beta(X_\beta) \subseteq \overline{\varrho^\alpha(\operatorname{Proj} \mathscr{X})}.$$

Of course, strongly reduced spectra $\mathscr{X}$ are reduced and satisfy $\operatorname{Proj}^+ \mathscr{X} = 0$. Our next result generalizes the fundamental theorem 3.2.1.

**Theorem 4.3.5** *Let $\mathscr{X}$ be a reduced locally convex $I$-spectrum consisting of Fréchet spaces. If $I$ has cardinality $\aleph_n$ then* $\operatorname{Proj}^k \mathscr{X} = 0$ *for $k \geq n+1$.*

**Proof.** The proof is almost the same as the one we chose to show 4.1.9. For $n = 0$ we use theorem 3.2.1 and then the induction runs as before if we choose $f : I \times I \longrightarrow I$ such that $\delta = f(\alpha, \beta)$ satisfies $\delta \geq \alpha, \delta \geq \beta$, and in addition, $\varrho^\alpha_\delta(X_\delta) \subseteq \overline{\varrho^\alpha_\gamma(X_\gamma)}$ and $\varrho^\beta_\delta(X_\delta) \subseteq \overline{\varrho^\beta_\gamma(X_\gamma)}$ for all $\gamma \in I$ with $\gamma \geq \delta$. □

We now want to calculate $\operatorname{Proj}^k \mathscr{X}$ for a locally convex $I$-spectrum representing a complete (LB)-space $X = \operatorname{Proj} \mathscr{X}$. We assume that $\mathscr{X}$ is strongly reduced and consists of Fréchet spaces. By corollary 4.3.4, $\operatorname{Proj}^k \mathscr{X}$ only depends on $X$ and it is thus enough to calculate $\operatorname{Proj}^k \mathscr{Y}$ for a particular strongly reduced spectrum $\mathscr{Y}$ with $\operatorname{Proj} \mathscr{Y} = X = \operatorname{ind} X_n$ where $X_n \subset X_{n+1}$ are Banach spaces with unit balls $B_n \subseteq B_{n+1}$. The canonical basis

$$J = \left\{ \Gamma \left( \bigcup_{n \in \mathbb{N}} \frac{1}{m_n} B_n \right) : (m_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}} \right\}$$

of $\mathscr{U}_0(X)$ has cardinality (at most) $|\mathbb{N}^{\mathbb{N}}|$, and defining $Y_U = \widetilde{(X, p_U)}$ for $U \in J$ we obtain a strongly reduced $J$-spectrum $\mathscr{Y}$ consisting of Banach spaces with $\operatorname{Proj} \mathscr{Y} = X$. In view of 4.3.5 it is not surprising that axioms about the cardinality of $\mathbb{N}^{\mathbb{N}}$ enter the game. If we assume the continuum hypothesis $|\mathbb{N}^{\mathbb{N}}| = \aleph_1$ we conclude $\operatorname{Proj}^k \mathscr{X} = 0$ for all $k \geq 2$. It remains to find $\operatorname{Proj}^1 \mathscr{X}$.

**Lemma 4.3.6** *If the directed set $I$ contains a smallest element $0 \in I$ and $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ is a projective $I$-spectrum with injective spectral maps $\varrho^\alpha_\beta$, then* $\operatorname{Proj}^1 \mathscr{X} = 0$ *if and only if for every $(x_\alpha)_{\alpha \in I} \in X_0^I$ with $x_\alpha - x_\beta \in \varrho^0_\alpha(X_\alpha)$ for all $\alpha \leq \beta$ there is $z \in X_0$ with $x_\alpha - z \in \varrho^0_\alpha(X_\alpha)$ for all $\alpha \in I$.*

**Proof.** We will use again the notation of theorem 4.1.3. If $\operatorname{Proj}^1 \mathscr{X} = 0$ and $(x_\alpha)_{\alpha \in I} \in X_0^I$ with $x_\alpha - x_\beta \in \varrho^0_\alpha(X_\alpha)$ for all $\alpha \leq \beta$ is given we choose $y = (y_{\alpha_0, \alpha_1})_{\alpha_0 \leq \alpha_1} \in \prod_{\alpha_0 \leq \alpha_1} X_{\alpha_0}$ with $\varrho^0_{\alpha_0}(y_{\alpha_0, \alpha_1}) = x_{\alpha_0} - x_{\alpha_1}$. Then $d_1(y) = 0$, hence there is $z = (z_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha$ with $d_0(z) = (\varrho^{\alpha_0}_{\alpha_1}(z_{\alpha_1}) - z_{\alpha_0})_{\alpha_0 \leq \alpha_1} = y$. We conclude $\varrho^0_{\alpha_0}(z_{\alpha_0}) - x_{\alpha_0} = \varrho^0_{\alpha_1}(z_{\alpha_1}) - x_{\alpha_1}$ for all $\alpha_0 \leq \alpha_1$ and $z = x_0 - z_0$ satisfies $x_\alpha - z \in \varrho^0_\alpha(X_\alpha)$ for all $\alpha \in I$. Let now the condition of the lemma be satisfied and $y \in \ker(d_1)$ be given. Then $(x_\alpha)_{\alpha \in I} = (\varrho^0_\alpha(y_{0,\alpha}))_{\alpha \in I} \in X_0^I$ satisfies $x_\alpha - x_\beta = -\varrho^0_\alpha(y_{\alpha,\beta}) \in \varrho^0_\alpha(X_\alpha)$ for all $\alpha \leq \beta$ and if we choose $z \in X_0$ and $z_\alpha \in X_\alpha$ with $(x_\alpha - z) = \varrho^0_\alpha(z_\alpha)$ we obtain $d_0((z_\alpha)_{\alpha \in I}) = y$. □

The following proposition is a variant of a result due to F.C. Schmerbeck [55]. We recall that if $\alpha, \beta$ are ordinal numbers then $\alpha < \beta$ if and only if $\alpha \in \beta$.

**Proposition 4.3.7** *Let $\omega$ be a limit ordinal and $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ be an $\omega$-spectrum such that* $\operatorname{Proj}^1 \mathscr{X}^\gamma = 0$ *for every $\gamma \in \omega$ where $\mathscr{X}^\gamma$ is the $\gamma$-spectrum $(X_\alpha, \varrho^\alpha_\beta)_{\alpha < \gamma}$. We further assume that all spectral maps $\varrho^\alpha_\beta$ are injective but* *not surjective, and that the cardinality of* $X_0$ *is less or equal than that of* $\omega$*. Then* $\operatorname{Proj}^1 \mathscr{X} \neq 0$.

**Proof.** Let $f : \omega \longrightarrow X_0$ be a surjective map. We use transfinite induction to construct a certain element $(x_\alpha)_{\alpha\in\omega} \in X_0^\omega$. For $\gamma \in \omega$ we assume that $(x_\alpha)_{\alpha<\gamma} \in X_0^\gamma$ is already constructed with

1. $\forall\, \alpha < \beta < \gamma \quad x_\alpha - x_\beta \in X_\alpha$ (which is considered as a subspace of $X_0$) and
2. $\forall\, \beta < \gamma$ with $\beta + 1 < \gamma \;\; \exists\, \alpha \leq \beta + 1 \quad x_\alpha - f(\beta) \notin X_\alpha$.

If $\gamma$ is a limit ordinal we use $\operatorname{Proj}^1 \mathscr{X}^\gamma = 0$ and lemma 4.3.6 to find $x_\gamma \in X_0$ with $x_\alpha - x_\gamma \in X_\alpha$ for all $\alpha < \gamma$. Then $(x_\alpha)_{\alpha<\gamma+1}$ still satisfies 1. and 2. since every $\beta < \gamma + 1$ with $\beta + 1 < \gamma + 1$ already satisfies $\beta + 1 < \gamma$ as $\gamma$ is a limit ordinal.

Let now $\gamma = \delta + 1$ be a successor ordinal. If there is $\alpha \leq \delta$ with $x_\alpha - f(\delta) \notin X_\alpha$ we set $x_\gamma = x_\delta$. Then $(x_\alpha)_{\alpha<\gamma+1}$ satisfies 1. and 2. Otherwise, we choose $r \in X_\delta \backslash X_{\delta+1}$ (which is possible since $\varrho^\delta_{\delta+1}$ is not surjective) and set $x_\gamma = r + f(\delta)$. For $\alpha < \gamma$ we have $x_\alpha - x_\gamma = (x_\alpha - f(\delta)) - r \in X_\alpha + X_\delta \subseteq X_\alpha$ which yields that 1. holds for $(x_\alpha)_{\alpha<\gamma+1}$, and also 2. holds since $x_\gamma - f(\delta) = r \notin X_\delta$.

Suppose now $\operatorname{Proj}^1 \mathscr{X} = 0$. Lemma 4.3.6 implies that there is $z \in X_0$ such that $x_\alpha - z \in X_\alpha$ for all $\alpha \in \omega$. Since $f$ is surjective there is $\beta \in \omega$ with $z = f(\beta)$, hence $x_\alpha - f(\beta) \in X_\alpha$ for all $\alpha \in \omega$, a contradiction to 2. □

We will now show that "most" complete (LB)-spaces can be represented as required in the proposition above. Recall that an (LB)-space (or, more generally, a (DF)-space) has the dual density condition (DDC) if its bounded sets are metrizable, for a detailed investigation of such spaces we refer to the work of Bierstedt and Bonet [6]. We call a directed set $I$ countably directed if each sequence $(\alpha_n)_{n\in\mathbb{N}} \in I^\mathbb{N}$ has an upper bound $\alpha \in I$ (i.e. $\alpha_n \leq \alpha$ for all $n \in \mathbb{N}$.)

**Lemma 4.3.8** *Let* $(X, \mathscr{S})$ *be a complete (DF)-space with the (DDC). Then there are a countably directed set* $I$ *and a strongly reduced projective* $I$*-spectrum* $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ *with Banach spaces* $X_\alpha$ *and injective spectral maps such that* $\operatorname{Proj} \mathscr{X} = (X, \mathscr{S})$*. If* $(X, \mathscr{S})$ *is not normed the spectral maps can be assumed to be non-surjective for* $\alpha \leq \beta, \alpha \neq \beta$.

**Proof.** Since $X$ has the countable neighbourhood property, i.e. for each sequence $(U_n)_{n\in\mathbb{N}}$ of 0-neighbourhoods in $X$ there are $U \in \mathscr{U}_0(X)$ and $S_n > 0$ with $U \subseteq S_n U_n$, it is easily seen that there is a coarser norm topology $\mathscr{T}$ on $X$ which induces the original topology on each bounded set. Let $\mathscr{A} = (A_n)_{n\in\mathbb{N}}$ be a fundamental sequence of bounded sets in $X$. According to [51, 8.1.12] the finest locally convex topology $\mathscr{T}(\mathscr{A})$ which coincides with $\mathscr{T}$ on all $A_n$ has a basis $\mathscr{U}$ consisting of absolutely convex sets of the form

$$\overline{\varepsilon_0 B} \cap \bigcap_{n\in\mathbb{N}} \overline{A_n \cap \varepsilon_n B}$$

which are closed and bounded in $(X, \mathscr{T})$ (where $B$ is the unit ball of the norm giving the topology $\mathscr{T}$ and the closures are taken with respect to $\mathscr{T}$). On the other hand, $\mathscr{T}(\mathscr{A}) = \mathscr{S}$ since $(X, \mathscr{S})$ is a (DF)-space [51, 8.3]. Given $U, V \in \mathscr{U}$ with $V \subseteq SU$ for some $S > 0$ we set $X_U = \widetilde{(X, p_U)}$ and denote by $\varrho_V^U$ the unique continuous extension of the identity map $(X, p_V) \longrightarrow (X, p_U)$. Since $V$ is closed in $\mathscr{T}$ it is closed in $(X, p_U)$, too. Hence $\varrho_V^U$ is injective by [39, §18.4(4)]. Let us say $U \sim V$ if $U \subseteq SV$ and $V \subseteq SU$ for some $S > 0$. We define $I = \mathscr{U}/\sim$ with the order $\alpha = [U] \leq [V] = \beta$ if $V \subseteq SU$ for some $S > 0$. Given $\alpha \in I$ we choose some $U$ with $\alpha = [U]$ and set $X_\alpha = X_U$ (as a locally convex space, $X_\alpha$ is independent of the representative), and for $\alpha = [U] \leq [V] = \beta$ we set $\varrho_\beta^\alpha = \varrho_V^U$. Then $I$ is countably directed because of the countable neighbourhood property and $\mathscr{X} = (X_\alpha, \varrho_\beta^\alpha)$ has the required properties.

Note that for $\alpha \leq \beta, \alpha \neq \beta$ the spectral maps $\varrho_\beta^\alpha$ are not surjective because of the open mapping theorem. □

**Theorem 4.3.9** *Let $\mathscr{X}$ be a strongly reduced locally convex projective $I$-spectrum consisting of Fréchet spaces such that* $\operatorname{Proj} \mathscr{X}$ *is a separable proper (DF)-space with the (DDC). Assuming the continuum hypothesis we then have* $\operatorname{Proj}^1 \mathscr{X} \neq 0$ *and* $\operatorname{Proj}^k \mathscr{X} = 0$ *for all* $k \geq 2$.

**Proof.** According to 4.3.4 it is enough to prove the result for a particular spectrum, hence we may assume that $\mathscr{X} = (X_\alpha, \varrho_\beta^\alpha)$ is as in lemma 4.3.8. Since $(X, \mathscr{S}) = \operatorname{Proj} \mathscr{X}$ is a proper (DF)-space, $I$ is not countable. On the other hand $I$ has some cofinal subset of cardinality $|\mathbb{N}^\mathbb{N}| = \aleph_1$. Let thus $\omega_1$ be the first uncountable ordinal and $(\alpha_\nu)_{\nu < \omega_1}$ be an enumeration of a cofinal subset of $I$. We construct an $\omega_1$-spectrum $\mathscr{Y} = (Y_\nu, \sigma_\mu^\nu)$ with the properties of proposition 4.3.7 by transfinite induction. We start with $Y_0 = X_{\alpha_0}$. If the $\nu$-spectrum $\mathscr{Y}^\nu$ is already constructed we set $Y_\nu = \operatorname{Proj} \mathscr{Y}^\nu$ if $\nu$ is an infinite limit ordinal and $Y_\nu = Y_\mu \cap X_\alpha$ if $\nu = \mu + 1$ is a successor ordinal where $\alpha \geq \alpha_\nu$ is chosen in such a way that $Y_\nu$ is properly contained in $Y_\mu$. With the obvious spectral maps, $\mathscr{Y} = (Y_\nu, \sigma_\mu^\nu)$ is a projective $\omega_1$-spectrum consisting of countable projective limits of Banach spaces, i.e. Fréchet spaces with injective and non-surjective spectral maps. $\mathscr{Y}^\nu$ is a countable and reduced spectrum of Fréchet spaces, hence satisfies $\operatorname{Proj}^1 \mathscr{Y}^\nu = 0$ for each $\nu < \omega_1$ by theorem 3.2.1. Finally, the cardinality of $Y_0 = X_{\alpha_0}$ is $\aleph_1 = |\omega_1|$ since $X_{\alpha_0}$ is separable. Proposition 4.3.7 yields $\operatorname{Proj}^1 \mathscr{Y} \neq 0$ and theorem 4.3.5 $\operatorname{Proj}^k \mathscr{Y} = 0$ for $k \geq 2$. □

**Corollary 4.3.10** *Let $X$ be a separable proper (DF)-space satisfying the (DDC). Assuming the continuum hypothesis we then have $\mathscr{C}^1(X) \neq 0$ and $\mathscr{C}^k(X) = 0$ for all $k \geq 2$.*

The corollary applies in particular to the space $\varphi$ of finite sequences endowed with the strongest locally convex topology. Moreover, it is easily seen that the property $\mathscr{C}^1(X) = 0$ is stable with respect to complemented subspaces. Under the continuum hypothesis we obtain $\mathscr{C}^1(X) \neq 0$ for every locally convex space $X$ which contains $\varphi$ as a complemented subspace, in particular, $\mathscr{C}^1(\mathscr{D}) \neq 0$. This answers problem 8 of [50, §12].

# 5

# The derived functors of Hom

In this chapter we use the derivatives of the Hom-functor to investigate the splitting of short exact sequences. This gives information about the existence of continuous linear solution operators. 5.1 contains the general theory for locally convex spaces, 5.2 is devoted to the case of Fréchet spaces, and in 5.3 it is shown that the space $\mathscr{D}'$ of distributions plays exactly the same role in the splitting theory for projective limits of (LS)-spaces as the space $s$ of rapidly decreasing sequences does for Fréchet spaces.

## 5.1 $\mathrm{Ext}^k$ in the category of locally convex spaces

Given a locally convex space $E$ we consider the functor $L(E,\cdot\,)$ assigning to a locally convex space $X$ the linear space $L(E,X)$ of continuous linear maps from $E$ to $X$ and to a morphism (= continuous linear map) $T: X \to Y$ the linear map $T^*: L(E,X) \to L(E,Y), f \mapsto T \circ f$. This covariant functor is additive and injective. Since the category of locally convex spaces has sufficiently many injective objects we can construct the derived functors $\mathrm{Ext}^k(E,\cdot\,)$ using injective resolutions. (The derivatives are traditionally denoted by $\mathrm{Ext}^k$ since they can be constructed as equivalence classes of extensions.)

**Theorem 5.1.1** *1. Let $E$ be a locally convex space and $0 \longrightarrow X \stackrel{I}{\longrightarrow} Y \stackrel{Q}{\longrightarrow} Z \longrightarrow 0$ an exact sequence of locally convex spaces. Then there is an exact complex*

$$0 \longrightarrow L(E,X) \stackrel{I^\star}{\longrightarrow} L(E,Y) \stackrel{Q^\star}{\longrightarrow} L(E,Z) \stackrel{\delta^\star}{\longrightarrow} \mathrm{Ext}^1(E,X) \longrightarrow \cdots$$

*2. Let $X$ be a locally convex space and $0 \longrightarrow E \stackrel{i}{\longrightarrow} F \stackrel{q}{\longrightarrow} G \longrightarrow 0$ an exact sequence of locally convex spaces. Then there is an exact complex*

$$0 \longrightarrow L(G,X) \stackrel{q_\star}{\longrightarrow} L(F,X) \stackrel{i_\star}{\longrightarrow} L(E,X) \stackrel{\delta_\star}{\longrightarrow} \mathrm{Ext}^1(G,X) \longrightarrow \cdots$$

*(where $q_*: L(G,X) \to L(F,X)$, $T \mapsto T \circ q$ etc.)*

J. Wengenroth: LNM 1810, pp. 77–107, 2003.

**Proof.** The first part is clear from the construction of $\mathrm{Ext}^k(E,\cdot\,)$. The second part can be proved by diagram chasing. Let us sketch the construction of the map $\delta_\star : L(E,X) \longrightarrow \mathrm{Ext}^1(G,X)$. We embed $X$ into an injective object $I_0$ an obtain a commutative diagram

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L(G,X) & \longrightarrow & L(G,I_0) & \longrightarrow & L(G,I_0/X) & \longrightarrow & \mathrm{Ext}^1(G,X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L(F,X) & \longrightarrow & L(F,I_0) & \longrightarrow & L(F,I_0/X) & \longrightarrow & \mathrm{Ext}^1(F,X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L(E,X) & \longrightarrow & L(E,I_0) & \longrightarrow & L(E,I_0/X) & \longrightarrow & \mathrm{Ext}^1(E,X) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}
$$

The rows are exact by the construction of $\mathrm{Ext}^1(G,X)$ and the columns are also exact (the second one since $I_0$ is injective). Now, the construction of the linear map $\delta_\star$ is canonical, and the maps $\mathrm{Ext}^k(E,X) \to \mathrm{Ext}^{k+1}(G,X)$ are constructed similarly. □

We will now present the connection between the vanishing of $\mathrm{Ext}^1(E,X)$ and splitting, extension, and lifting problems. For later use we do this in a slightly more general setting.

**Definition 5.1.2** *A semi-abelian category is quasi-abelian if for each exact sequence* $0 \longrightarrow X \stackrel{i}{\longrightarrow} Y \stackrel{q}{\longrightarrow} Z \longrightarrow 0$ *the following conditions hold.*

1. *For each $T : X \to E$ there is a commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle T} & & \uparrow{\scriptstyle \widetilde{T}} & & \Vert & & \\
0 & \longrightarrow & X & \stackrel{i}{\longrightarrow} & Y & \stackrel{q}{\longrightarrow} & Z & \longrightarrow & 0
\end{array}
$$

   *with exact rows (this diagram is called push-out of $T$).*
2. *For each $R : H \to Z$ there is a commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \stackrel{i}{\longrightarrow} & Y & \stackrel{q}{\longrightarrow} & Z & \longrightarrow & 0 \\
 & & \Vert & & \uparrow{\scriptstyle \widetilde{R}} & & \uparrow{\scriptstyle R} & & \\
0 & \longrightarrow & X & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0
\end{array}
$$

   *with exact rows (this diagram is called pull-back of $R$).*

It is easily seen that the category of locally convex spaces is quasi-abelian, for the push-out one takes $F = (E \times Y)/L$ with $L = \{(T(x), i(x)) : x \in X\}$, and for the pull back $G = \{(h, y) \in H \times Y : R(h) = q(y)\}$. In this case, one can easily check that if $T$ is a topological embedding then so is $\widetilde{T}$ and the cokernels of $T$ and $\widetilde{T}$ are isomorphic, and if $R$ is a quotient map then so is $\widetilde{R}$ and the kernels of $R$ and $\widetilde{R}$ are isomorphic.

**Proposition 5.1.3** *For two objects $E, X$ of a quasi-abelian category the following conditions are equivalent.*

1. *Every exact sequence*

$$0 \longrightarrow X \stackrel{i}{\longrightarrow} Y \stackrel{q}{\longrightarrow} E \longrightarrow 0$$

*splits (i.e. $q$ has a right inverse or equivalently, $i$ has a left inverse).*

2. *For each exact sequence*

$$0 \longrightarrow X \longrightarrow Y \stackrel{q}{\longrightarrow} Z \longrightarrow 0$$

*and every $R : E \to Z$ there is a lifting $\widetilde{R} : E \to Y$, i.e. $q \circ \widetilde{R} = R$.*

3. *For each exact sequence*

$$0 \longrightarrow C \stackrel{i}{\longrightarrow} D \longrightarrow E \longrightarrow 0$$

*and every $T : C \to X$ there is an extension $\widetilde{T} : D \to X$, i.e. $\widetilde{T} \circ i = T$.*

*In the category of locally convex spaces each of these conditions is equivalent to* $\mathrm{Ext}^1(E, X) = 0$.

**Proof.** The equivalence of 1. and 2., and 1. and 3. is easily seen using pull back and push out. The equivalence of 2. and $\mathrm{Ext}^1(E, X)$ comes from the first part of 5.1.1. □

If $X$ is a Fréchet space there is a close connection between $\mathrm{Ext}^k(E, X)$ and the derived functors of Proj.

**Definition 5.1.4** *A reduced countable projective spectrum $\mathscr{X} = (X_n, \varrho_m^n)$ of Banach spaces is representing for a Fréchet space $X$ if* $\mathrm{Proj}\, \mathscr{X} \cong X$.

*A Fréchet space $X$ is called locally $E$-acyclic (for a locally convex space $E$) if $X$ has a representing spectrum $\mathscr{X} = (X_n, \varrho_m^n)$ such that* $\mathrm{Ext}^k(E, X_n) = 0$ *for all $n \in \mathbb{N}$ and $k \geq 1$.*

Clearly, every Fréchet space has a representing spectrum and all representing spectra are equivalent. Moreover, a representing spectrum $\mathscr{X}$ satisfies $\mathrm{Proj}^1 \mathscr{X} = 0$ and $\mathrm{Proj}^+ \mathscr{X} = 0$ since it is reduced.

If $E$ is projective limit of a strongly reduced (not necessarily countable) spectrum of projective Banach spaces, it is easy to see that every Fréchet space $X$ is locally $E$ acyclic. In particular, this applies to complete nuclear spaces $E$.

**Proposition 5.1.5** *Let $E$ be a locally convex space and $X$ a Fréchet space with a representing spectrum $\mathscr{X} = (X_n, \varrho_m^n)$.*

1. $\operatorname{Ext}^1(E, X) = 0$ *implies* $\operatorname{Proj}^1 \mathscr{Y} = 0$ *where* $\mathscr{Y} = (L(E, X_n), R_m^n)$ *with* $R_m^n(T) = \varrho_m^n \circ T$.
2. *If $X$ is locally $E$-acyclic then* $\operatorname{Ext}^1(E, X) \cong \operatorname{Proj}^1 \mathscr{Y}$ *(with $\mathscr{Y}$ as above) and* $\operatorname{Ext}^k(E, X) = 0$ *for every* $k \geq 2$.

**Proof.** The sequence

$$0 \longrightarrow X \longrightarrow \prod_{n \in \mathbb{N}} X_n \longrightarrow \prod_{n \in \mathbb{N}} X_n \longrightarrow 0$$

is exact in the category of locally convex spaces. Applying $L(E, \cdot)$ gives an exact sequence

$$0 \longrightarrow L(E, X) \longrightarrow \prod_{n \in \mathbb{N}} L(E, X_n) \longrightarrow \prod_{n \in \mathbb{N}} L(E, X_n) \longrightarrow$$

$$\longrightarrow \operatorname{Ext}^1(E, X) \longrightarrow \prod_{n \in \mathbb{N}} \operatorname{Ext}^1(E, X_n) \longrightarrow \cdots$$

This easily implies the assertions. □

**Lemma 5.1.6** *Let $E$ be a locally convex space and $X$ a Fréchet space with representing spectrum $\mathscr{X} = (X_n, \varrho_m^n)$. If either $E$ or $X$ is nuclear, then $\mathscr{Y} = (L_b(E, X_n), R_m^n)$ is strongly reduced (where $R_m^n(T) = \varrho_m^n \circ T$ and $L(E, X_n)$ is endowed with the topology of uniform convergence on the bounded subsets of $E$).*

**Proof.** Let first $X$ be nuclear. By passing to an equivalent representing spectrum for $X$ (which causes a change to an equivalent spectrum for $\mathscr{Y}$) we may assume $X_n \cong \ell^\infty$ and that $\varrho_{n+1}^n$ are all nuclear. Given $n \in \mathbb{N}$ we choose $m \geq n$ such that $\varrho_m^n(X_m) \subseteq \overline{\varrho^n(X)}$ (which is possible since reduced spectra of Banach spaces are strongly reduced). Since $\varrho_{m+1}^m$ is nuclear there are $\varphi_j \in X'_{m+1}$ and $x_j \in X_m$ with $\sum_{j=1}^{\infty} \|\varphi_j\|_{m+1}^* \|x_j\|_m < \infty$ (where $\|\ \|_m$ is the norm of $X_m$ and $\|\ \|_{m+1}^*$ the dual norm on $X'_{m+1}$) such that $\varrho_{m+1}^m = \sum_{j=1}^{\infty} \varphi_j \otimes x_j$ (where $\varphi_j \otimes x_j$ is the one-dimensional operator $\xi \mapsto \varphi_j(\xi) x_j$). Let now $T \in L(E, X_{m+1})$ and $U = \Gamma(U) \in \mathscr{U}_0(L_b(E, X_n))$ be given. We obtain $R_{m+1}^m(T) = \sum_{j=1}^{\infty} (\varphi_j \circ T) \otimes x_j$ and $R_{m+1}^n(T) = \sum_{j=1}^{\infty} (\varphi_j \circ T) \otimes \varrho_m^n(x_j)$ where the second series converges in $L_b(E, X_n)$. Hence there is $l \in \mathbb{N}$ such that $\sum_{j=l+1}^{\infty} (\varphi_j \circ T) \otimes \varrho_m^n(x_j) \in \frac{1}{2} U$. Moreover, there are $y_j \in X$ such that $\sum_{j=1}^{l} (\varphi_j \circ T) \otimes (\varrho_m^n(x_j) - \varrho^n(y_j)) \in \frac{1}{2} U$. Defining

$S = \sum_{j=1}^{l} (\varphi_j \circ T) \otimes y_j \in L(E,X) = \mathrm{Proj}\,\mathscr{Y}$ we obtain $R^n_m(T) - R^n(S) \in U$, hence $\mathscr{Y}$ is strongly reduced.

If $E$ is nuclear, the proof is similar. Given $n \in \mathbb{N}$ we choose $m \geq n$ as above. For any $T \in L(E, X_m)$ and $U = \Gamma(U) \in \mathscr{U}_0(E, X_n)$ we can find a continuous seminorm $q$ on $E$ such that $T$ factorizes as a nuclear operator over the completion $E_q$ of $(E,q)/\ker q$, i.e. there are a nuclear operator $\widetilde{T} : E_q \to X_m$ and $\pi : E \to E_q$ with $\widetilde{T} \circ \pi = T$. There are $\varphi_j \in E'_q$ and $x_j \in X_m$ with $\widetilde{T} = \sum_{j=1}^{\infty} \varphi_j \otimes x_j$ and now, the same argument as above produces an $S \in L(E,X)$ with $R^n_m(T) - R^n(S) \in U$. □

As a consequence, we obtain the following result of Palamodov [50, theorem 9.1].

**Theorem 5.1.7** *Let $E$ be a (DF)-space and $X$ a Fréchet space such that one of them is nuclear. Then* $\mathrm{Ext}^k(E,X) = 0$ *for all* $k \in \mathbb{N}$.

**Proof.** We represent $X = \mathrm{Proj}\,\mathscr{X}$ as the limit of a reduced spectrum of Banach spaces. Since $E$ is a (DF)-space, the spectrum $\mathscr{Y}$ from proposition 5.1.6 is formed by Fréchet spaces $L_b(E, X_n)$. Theorem 3.2.1 implies $\mathrm{Ext}^k(E,X) = \mathrm{Proj}^k \mathscr{Y} = 0$ for all $k \in \mathbb{N}$. □

Palamodov conjectured "it is natural to expect that the following proposition 'dual' to theorem [5.1.7] is valid: $\mathrm{Ext}^i(E,X) = 0$, $i \geq 1$, if $E$ is metric, $X$ is a complete dual-metric space, and one of them is a nuclear space". The obstacle which one meets is that there is seemingly no way to apply results about the countable Proj functor to the problem, e.g. there is no canonical $E$-acyclic representation $X = \mathrm{Proj}\,\mathscr{X}$ as a countable projective limit. The only Fréchet space for which we can provide a partial *positive* answer to Palamodov's conjecture is $E = \omega = \mathbb{K}^{\mathbb{N}}$. The following result was obtained jointly with L. Frerick.

**Proposition 5.1.8** *Let $X$ be a complete (DF)-space with the (DDC). Then* $\mathrm{Ext}^1(\omega, X) = 0$.

**Proof.** According to 4.3.8 we can represent $X = \mathrm{Proj}\,\mathscr{X}$, where $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ is a strongly reduced $I$-spectrum consisting of Banach spaces and injective spectral maps with a countably directed set $I$. Let $i : X \to \prod_{\alpha \in I} X_\alpha$, $x \mapsto (\varrho^\alpha x)_{\alpha \in I}$ be the canonical embedding and $q : \prod_{\alpha \in I} X_\alpha \longrightarrow Z := \left(\prod_{\alpha \in I} X_\alpha\right)/i(X)$ the corresponding quotient map. Since $\mathrm{Ext}^1(\omega, \prod_{\alpha \in I} X_\alpha) = 0$ we obtain the exact sequence

$$0 \longrightarrow L(\omega, X) \longrightarrow L\left(\omega, \prod_{\alpha \in I} X_\alpha\right) \xrightarrow{q^\star} L(\omega, Z) \longrightarrow \mathrm{Ext}^1(\omega, X) \longrightarrow 0.$$

Hence, we have to show that $q^* : L\left(\omega, \prod_{\alpha\in I} X_\alpha\right) \longrightarrow L(\omega, Z), S \mapsto q\circ S$ is surjective. Let therefore $T \in L(\omega, Z)$ be given and denote the unit vectors in $\omega$ by $e_n$. Then there are $x^n = (x^n_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha$ with $q(x^n) = T(e_n)$. Let $B_\alpha$ be the unit ball of $X_\alpha$. Since $q$ is open and $T$ is continuous we obtain

$$\forall\, E \subset I \text{ finite } \ \exists\, n \in \mathbb{N} \quad T\Big(\Big[\{e_k : k \geq n\}\Big]\Big) \subseteq q\left(\prod_{\alpha\in E} B_\alpha \times \prod_{\alpha\in I\setminus E} X_\alpha\right)$$

(where again, $[A]$ denotes the liner hull of a set $A$). As the left hand side of this inclusion is a linear space we can multiply with an arbitrary $\delta > 0$ without changing the left side and thus obtain $T(e_k) \in \delta q(\prod_{\alpha\in I} B_\alpha \times \prod_{\alpha\in I\setminus E} X_\alpha)$ for $k \geq n$ and $\delta > 0$, and therefore

$$\forall\, k \geq n, \delta > 0\ \exists\, z = z(\delta,k) \in X\ \forall\, \alpha \in E \quad x^k_\alpha - \varrho^\alpha(z) \in \delta B_\alpha.$$

For $\alpha$, $\beta \in E$ with $\alpha \leq \beta$ we thus get

$$\begin{aligned}\varrho^\alpha_\beta(x^k_\beta) - x^k_\alpha &= \varrho^\alpha_\beta\left(x^k_\beta - \varrho^\beta(z(\delta,k))\right) - \left(x_\alpha - \varrho^\alpha(z(\delta,k))\right)\\ &\in \bigcap_{\delta>0}\Big(\varrho^\alpha_\beta(\delta B_\beta) + \delta B_\alpha\Big) = \{0\}.\end{aligned}$$

We have proved:

$$(*)\ \forall\, E \subset I \text{ finite } \ \exists\, n \in \mathbb{N}\ \forall\, k \geq n,\ \alpha,\beta \in E,\ \alpha \leq \beta : \quad \varrho^\alpha_\beta(x^k_\beta) = x^k_\alpha.$$

We may assume that $I$ has a smallest element $0 \in I$ since otherwise we can consider $\widetilde{I} = \{\alpha \in I : \alpha \geq \alpha_0\}$ for some $\alpha_0 \in I$. For $n \in \mathbb{N}$ we define

$$I_n = \left\{\alpha \in I :\ \varrho^0_\alpha x^k_\alpha = x^k_0 \text{ for all } k \geq n\right\}.$$

Applying $(*)$ to $\{0, \alpha\}$ we obtain $\bigcup_{n\in\mathbb{N}} I_n = I$. And since $I$ is countably directed we find an $n_0 \in \mathbb{N}$ such that $I_{n_0}$ is cofinal in $I$ (i.e. for all $\alpha \in I$ there is $\beta \in I_{n_0}$ with $\alpha \leq \beta$) since otherwise there would be $\alpha_n \in I$ not dominated by any $\beta \in I_n$, and considering an $\alpha \in I$ with $\alpha_n \leq \alpha$ for all $n \in \mathbb{N}$ would yield a contradiction to $I = \bigcup_{n\in\mathbb{N}} I_n$. Since $I_{n_0}$ is cofinal, the canonical map

$$X \to \operatorname{Proj} \widetilde{\mathscr{X}},\ x \mapsto (\varrho^\alpha x)_{\alpha\in I_{n_0}}$$

(where $\widetilde{\mathscr{X}}$ is the restricted $I_{n_0}$-spectrum), is an isomorphism. Given $k \geq n_0$ and $\alpha, \beta \in I_{n_0}$ with $\alpha \leq \beta$ the injectivity of $\varrho^0_\alpha$ yields $\varrho^\alpha_\beta(x^k_\beta) = x^k_\alpha$ since $\varrho^0_\alpha\left(\varrho^\alpha_\beta(x^k_\beta) - x^k_\alpha\right) = \varrho^0_\beta(x^k_\beta) - \varrho^0_\alpha(x^k_\alpha) = 0$. Hence, there are $z^k \in X$ with $\varrho^\alpha(z^k) = x^k_\alpha$ for all $k \geq n_0$ and $\alpha \in I_{n_0}$. We now set $z^k = 0$ for $1 \leq k < n_0$ and $\widetilde{T}(e_k) := \left(x^k_\alpha - \varrho^\alpha(z^k)\right)_{\alpha\in I}$ for $k \in \mathbb{N}$, and we define

$$\widetilde{T} : L = \Big[\{e_k : k \in \mathbb{N}\}\Big] \longrightarrow \prod_{\alpha \in I} X_\alpha$$

by linear extension. We show that $\widetilde{T}$ is continuous if $L$ is endowed with the topology induced by $\omega$. Given $\alpha \in I$ we have to prove that $\pi_\alpha \circ \widetilde{T} : L \to X_\alpha$ is continuous where $\pi_\alpha$ is the canonical projection. There is $n \geq n_0$ such that $\varrho^0_\alpha(x^k_\alpha) = x^k_0$ for all $k \geq n$, and thus the injectivity of $\varrho^0_\alpha$ yields $x^k_\alpha = \varrho^\alpha(z^k)$, since $\varrho^0_\alpha\big(x^k_\alpha - \varrho^\alpha(z^k)\big) = x^k_0 - x^k_0 = 0$ for $k \geq n$. This implies the continuity of $\pi_\alpha \circ \widetilde{T}$.

Since $L$ is dense in $\omega$ we can finally extend $\widetilde{T}$ continuously to $S : \omega \to \prod_{\alpha \in I} X_\alpha$ and obtain $q \circ S = T$ since both operators are continuous and coincide on the dense subspace $L$. □

**Remark 5.1.9** 1. Let $E$ be a topological subspace of $\omega$ containing the unit vectors and $X$ a complete (DF)-space with the (DDC). The proof of 5.1.8 then shows $\mathrm{Ext}^1(E, X) = 0$.

2. Let now $E = \Phi$ be the space of finite sequences endowed with the topology of pointwise convergence, i.e. the relative topology induced by $\omega$. Then $\mathrm{Ext}^1(\Phi, X) = 0$ is related to a fairly weak completeness property of quotients $Y/X$. To see this we introduce a notion which also plays a significant role in Vogt's investigation of splitting theorems for Fréchet spaces in absence of continuous norms [66].

**Definition 5.1.10** *For a locally convex space $(X, \mathscr{T})$ we denote by $\mathscr{T}^{sk}$ the group topology on $X$ having $\{\ker(p) : p \in cs(X, \mathscr{T})\}$ as a basis of the 0-neighbourhood filter. $\mathscr{T}^{sk}$ is called associated seminorm–kernel topology.*

*$(X, \mathscr{T})$ is (sequentially) sk-complete if $(X, \mathscr{T}^{sk})$ is a (sequentially) complete topological group.*

Considered as a functor acting on the category of locally convex spaces with values in the category of abelian topological groups, the assignment of the associated seminorm–kernel topology is of injective type, but we will not investigate its homological properties. We have the following simple result.

**Proposition 5.1.11** *Let $\Phi$ be the space of finite sequences endowed with the relative topology of $\omega$.*

1. *A locally convex space $X$ is sequentially sk-complete if and only if every continuous linear operator $T : \Phi \to X$ has an extension $\widetilde{T} : \omega \to X$.*
2. *If $0 \longrightarrow X \longrightarrow Y \stackrel{q}{\longrightarrow} Z \longrightarrow 0$ is an exact sequence of locally convex spaces and $Y$ is sequentially sk-complete then $\mathrm{Ext}^1(\Phi, X) = 0$ implies that $Z$ is sequentially sk-complete.*

**Proof.** Let $\mathscr{T}_\omega$ be the topology of $\omega = \prod_{n \in \mathbb{N}} \mathbb{K}$. Then $\mathscr{T}^{sk}_\omega$ is the topology of $\prod_{n \in \mathbb{N}} \mathbb{K}_d$ where $\mathbb{K}_d$ is the field endowed with the discrete topology. Let now

$(X, \mathscr{T})$ be sequentially $sk$-complete and $T : \Phi \to (X, \mathscr{T})$ a continuous linear map. Since $T$ is also continuous as a map $\big(\Phi, \mathscr{T}_\omega^{sk} \cap \Phi\big) \to (X, \mathscr{T}^{sk})$ and $\Phi$ is dense in $(\omega, \mathscr{T}_\omega^{sk})$ there is a unique continuous extension $\widetilde{T} : (\omega, \mathscr{T}_\omega^{sk}) \to (X, \mathscr{T}^{sk})$ which is easily seen to be linear and continuous as a map $(\omega, \mathscr{T}_\omega) \to (X, \mathscr{T})$.

If on the other hand $(X, \mathscr{T})$ has the extension property as in the lemma and $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, \mathscr{T}^{sk})$ we define $T : \Phi \to X$ by $T(e_n) = x_n - x_{n-1}$ (where $e_n$ are the unit vectors and $x_0 = 0$) and linear extension. Then $T$ is continuous and if $\widetilde{T} : \omega \to X$ is the continuous extension it is easily seen that $(x_n)_{n\in\mathbb{N}}$ converges in $(X, \mathscr{T}^{sk})$ to $x = \widetilde{T}(\sum_{n=1}^{\infty} e_n)$.

To prove the second part, let $T : \Phi \to Z$ be a continuous linear operator. Since $\operatorname{Ext}^1(\Phi, X) = 0$ the second part of proposition 5.1.3 implies that there is a continuous linear $S : \Phi \to Y$ with $q \circ S = T$. If $\widetilde{S}$ is an extension of $S$ then $q \circ \widetilde{S}$ is an extension of $T$, and part 1. of the proposition implies that $Z$ is sequentially $sk$-complete. □

**Proposition 5.1.12** *Let $X$ be a separated sequentially $sk$-complete locally convex space with* $\operatorname{Ext}^1(\omega, X) = 0$ . *Then $X$ is complete.*

**Proof.** If $X$ is incomplete we can find $x \in \widetilde{X} \backslash X$ which gives rise to an exact sequence

$$0 \longrightarrow X \stackrel{i}{\longrightarrow} Y \stackrel{q}{\longrightarrow} \mathbb{K}_0 \longrightarrow 0$$

where $Y = X + [x]$ is endowed with the relative topology of $\widetilde{X}$ and $\mathbb{K}_0$ is the filed $\mathbb{K}$ endowed with coarsest topology $\{\emptyset, \mathbb{K}\}$. The sequence is exact since $X$ is dense in $Y$. By considering a Hamel basis of $\omega$ containing the unit vectors it is easy to find a linear non-zero operator $T : \omega \to \mathbb{K}$ with $T|_\Phi = 0$. Then $T : \omega \to \mathbb{K}_0$ is continuous and proposition 5.1.3 implies that there is a lifting $S \in L(\omega, Y)$ with $q \circ S = T$. Since $q \circ S(\Phi) = \{0\}$ there is $R \in L(\Phi, X)$ with $i \circ R = S|_\Phi$ (where again $\Phi$ is considered as as topological subspace of $\omega$). Let $\widetilde{R} \in L(\omega, X)$ be the extension of $R$. Then $i \circ \widetilde{R}$ and $S$ are continuous operators on $\omega$ which coincide on the dense subspace $\Phi$, hence $i \circ \widetilde{R} = S$ because $Y$ is separated. This yields $T = q \circ S = q \circ i \circ \widetilde{R} = 0$, contradicting the choice of $T$. □

Combining this with the results of chapter 4 we obtain a partial answer to Palamodov's question mentioned above.

**Proposition 5.1.13** *Let $\varphi$ be the space of finite sequences endowed with the finest locally convex topology. Assuming the continuum hypothesis we have* $\operatorname{Ext}^2(\omega, \varphi) \neq 0$.

**Proof.** We represent $\varphi$ (which is certainly a complete (DF)-space with the (DDC)) as a projective limit as in the proof of 5.1.8 and obtain the exact sequence

$$0 \longrightarrow \varphi \longrightarrow \prod_{\alpha \in I} X_\alpha \longrightarrow Z \longrightarrow 0$$

This implies $\mathrm{Ext}^2(\omega, \varphi) = \mathrm{Ext}^1(\omega, Z)$ since $\mathrm{Ext}^k(\omega, \prod_{\alpha \in I} X_\alpha) = 0$ for all $k \in \mathbb{N}$. Let again $\Phi$ be the space $\varphi$ with relative topology of $\omega$. Then $\mathrm{Ext}^1(\Phi, \varphi) = 0$ by the first part of 5.1.9 and the second part of 5.1.11 implies that $Z$ is sequentially $sk$-complete. On the other hand, $Z$ is incomplete by 4.3.9, and 5.1.12 yields $\mathrm{Ext}^1(\omega, Z) \neq 0$. □

The space $\omega$ is the only *Fréchet* space to which the techniques developed here apply. However, Palamodov's question as stated is completely solved (under the continuum hypothesis) in [68]. There is a *normed* space $E$ such that $\mathrm{Ext}^1(E, X) \neq 0$ for each infinite dimensional nuclear (DF)-space $X$. Moreover, for nuclear (DF)-spaces $X$ the higher derivatives $\mathrm{Ext}^k(E, X)$ all vanish for $k \geq 3$ and each locally convex space $E$.

Let us make one more remark about $\mathrm{Ext}^1(E, X)$ where $E$ is a nuclear Fréchet space and $X$ a nuclear (LB)-space. According to 5.1.3 $\mathrm{Ext}^1(E, X) = 0$ is equivalent to the splitting of all exact sequences

$$0 \longrightarrow X \longrightarrow Y \longrightarrow E \longrightarrow 0.$$

If $Y$ has some reasonable properties, e.g. if $Y$ is an (LF)-space or a strongly reduced projective limit of (LB)-space, then one can indeed show by ad hoc methods that such an exact sequence splits. There is however no general "three space theorem" which allows to conclude that $Y$ is an (LF)-space or a projective limit of (LB)-spaces, hence the general problem remains open.

In view of applications to splitting results, assumptions about the space $Y$ are harmless: $Y$ is the domain of the operator under consideration and generally known. In the following sections we will therefore investigate splitting results in smaller categories.

## 5.2 Splitting theory for Fréchet spaces

This topic has quite a long tradition in what is called structure theory of Fréchet spaces. We will present here only one theorem which is a variant of a characterization due to Vogt [63] and Frerick and the author [29] where results about the derived projective limit functor are applied. The case of sequence spaces was solved by J. Krone and Vogt [40].

Let $E$ and $F$ be Fréchet spaces. We represent $F$ as the limit of a reduced projective spectrum $\mathscr{F} = (F_n, \varrho_m^n)$ consisting of Banach spaces $(F_n, \|\cdot\|_n)$. If we confine ourselves to the case where $F$ is locally $E$-acyclic we obtain from proposition 5.1.5 that $Ext^1(E,F) = \mathrm{Proj}^1(L(E,F_n), R_m^n)$ and $Ext^k(E,F) = 0$ for $k \geq 2$. The spaces $Y_n = L(E,F_n)$ can be endowed with (LB)-space topologies (the associated bornological topologies to the topologies of uniform convergence on the bounded subsets of $E$), hence the results of section 3.2 lead to characterizations of $Ext^1(E,F) = 0$. For instance, theorem 3.2.16 immediately gives $Ext^1(E,F) = 0$ if and only if

$$\forall\, n \in \mathbb{N}\ \exists\, U_n \in \mathscr{U}_0(E),\ m \geq n\ \forall\, T \in L(E,F_m)\ \exists\, S \in L(E,F)$$

$$\sup\{\|(\varrho_m^n \circ T - \varrho^n \circ S)(x)\|_n : x \in U_n\} \leq 1.$$

This characterization is not yet easily applicable since it requires a very strong approximation of all operators from $E$ to $F_m$. However, in several cases the conditions from theorem 3.2.14 can be turned into a set of inequalities which allows evaluation in concrete cases. The conditions below are variants of notions introduced by Vogt [63].

**Definition 5.2.1** *A pair $(E,F)$ of Fréchet spaces satisfies $(S_3^*)$ if there are fundamental systems $(|\cdot|_N)_{N\in\mathbb{N}}$ and $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of seminorms for $E$ and $F$, respectively, such that*

$$\forall\ n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\, N \in \mathbb{N}\ \forall\, M \geq N\ \exists\, K \in \mathbb{N}, S > 0$$

$$\forall\, x \in E_K,\ f \in F_m^* \quad |\sigma_K^M(x)|_M \|f\|_m^* \leq S\Big(|x|_K \|f\|_k^* + |\sigma_K^N(x)|_N \|f\|_n^*\Big)$$

*where $E_K$ is the Hausdorff completion of $(E, |\cdot|_K)$, $\sigma_K^M$ are the canonical maps, and $F_m^* = \{f \in F' : \|f\|_m^* := \sup\{|f(y)| : \|y\|_m \leq 1\} < \infty\}$.*

*If these inequalities only hold for $f \in F_n^*$ the condition is called $(S_3^\bullet)$.*

Note that the "dual norm" $\|f\|_n^*$ takes the value $+\infty$ if $f$ does not belong to $F_n^*$. If $\sigma_K^N(x) \neq 0$ the inequality in $(S_3^*)$ is then trivial.

Vogt [63] used a similar condition $(S_2^*)$ where $N$ should be independent of $k$. Since the importance of these conditions is mainly due to their sufficiency for $\mathrm{Ext}^1(E,F) = 0$ we prefer to formulate the results with $(S_3^*)$. A posteriori, both conditions are equivalent.

It is easy to see that $(S_3^*)$ as well as $(S_3^\bullet)$ are invariant under a change of the seminorm system for $E$, one can even take any representing spectrum

$\mathscr{E} = (E_N, \sigma_M^N)$ for $E$. The same holds for $(S_3^\bullet)$ if the seminorms (or the representing spectrum) of $F$ are changed, but contrary to a claim in [63, page 175] the analogous remark does not apply to $(S_3^*)$ or $(S_2^*)$ as the example in 5.2.3 below shows. Thats why one has to formulate $(S_3^*)$ with the existence quantifier for the seminorms.

Let us recall that a Fréchet space $X$ is countably normed (a quojection) if it has a representing spectrum $\mathscr{X} = (X_n, \varrho_m^n)$ with injective (surjective) spectral maps. Note that passing to another representing spectrum may destroy the injectivity (surjectivity) of the spectral maps. The following result is contained in [63].

**Proposition 5.2.2** *Let $(E, F)$ be a pair of Fréchet spaces.*

1. $\mathrm{Ext}^1(E, F) = 0$ *implies that $(E, F)$ satisfies $(S_3^*)$.*
2. *If $E$ is not countably normed, then $(S_3^*)$ holds if and only if $F$ is a quojection.*
3. $\mathrm{Ext}^1(E, F) = 0$ *holds for every quojection $F$ which is locally $E$-acyclic.*
4. *If $E$ is countably normed, the pair $(E, F)$ satisfies $(S_3^*)$ if (and only if) it satisfies $(S_3^\bullet)$.*

**Proof.** We begin with some general considerations. Let $\mathscr{F} = (F_n, \varrho_m^n)$ be the representing spectrum for $F$ given by the Hausdorff completions $F_n$ of $(F, \|\cdot\|_n)$ and the canonical maps. Assuming $\mathrm{Ext}^1(E, F) = 0$, proposition 5.1.5 implies $\mathrm{Proj}^1(L(E, F_n), R_m^n) = 0$ where $R_m^n(T) = \varrho_m^n \circ T$. Since $L(E, F_n)$ can be endowed with the (LB)-space topology $\mathrm{ind}_N\,[B_{n,N}]$ with

$$B_{n,N} = \{T \in L(E, F_n) : \sup\{\|Tx\|_n : x \in U_N\} \leq 1\}$$

where $(U_N)_{N\in\mathbb{N}}$ is a basis of $\mathscr{U}_0(E)$, the remark following theorem 3.2.18 tells us that the spectrum $(L(E, F_n), R_m^n)$ satisfies $(P_3)$, i.e.

$$\forall\; n \in \mathbb{N}\; \exists\, m \geq n\; \forall\, k \geq m\; \exists\, N \in \mathbb{N}\; \forall\, M \geq N\; \exists\, K \in \mathbb{N}, S > 0$$

$$R_m^n(B_{m,M}) \subseteq S\Big(R_k^n(B_{k,K}) + B_{n,N}\Big).$$

According to [63] we call this condition $(S_3)$. Let us show that it implies the following property $(\widetilde{S}_3)$ :

$$\forall\; n \in \mathbb{N}\; \exists\, m \geq n\; \forall\, k \geq m\; \exists\, N \in \mathbb{N}\; \forall\, M \geq N\; \exists\, K \in \mathbb{N}, S > 0\; \forall\, x \in E_K$$

$$|\sigma_K^M(x)|_M\, \varrho_m^n(V_m) \subseteq S\Big(|x|_K\, \varrho_k^n(V_k) + |\sigma_K^N(x)|_N\, V_n\Big)$$

where $\mathscr{E} = (E_N, \sigma_M^N)$ is any representing spectrum for $E$ and $V_n$ is the closed unit ball of $F_n$. Of course, we may assume

$$U_N = \{x \in E : |\sigma^N(x)|_N \leq 1\}.$$

Let $n \leq m \leq k$, $N \leq M \leq K$ and $S > 0$ be as in $(S_3)$. We may assume that $\sigma^K(E)$ is dense in $E_K$ (otherwise we prove condition$(\widetilde{S}_3)$ for some $\widetilde{K} \geq K$

such that $\sigma^K(E)$ is dense in $\sigma^K_{\widetilde{K}}\left(E_{\widetilde{K}}\right))$. Given $x \in E_K$ and $y \in V_m$ we choose $\varphi \in E'_M$ with $|\varphi|^*_M \le 1$ and $\varphi\left(\sigma^M_K(x)\right) = |\sigma^M_K(x)|_M$. We define $T \in L(E, F_m)$ by $T(\xi) = \varphi\left(\sigma^M(\xi)\right) y$ and obtain $T \in B_{m,M}$. Hence there are $P \in B_{k,K}$ and $Q \in B_{n,N}$ with $\varrho^n_m \circ T = S(\varrho^n_k \circ P + Q)$. Now, we choose a sequence $\xi_l$ in $E$ with $\sigma^K(\xi_l) \to x$ in $E_K$ and obtain

$$|\sigma^M_K(x)|_M \varrho^n_m(y) = \lim_{l\to\infty} \varrho^n_m \circ T(\xi_l) = S\left(\lim_{l\to\infty} \varrho^n_k \circ P(\xi_l) + \lim_{l\to\infty} Q(\xi_l)\right)$$

where all limits exist since $T$, $P$ and $Q$ are continuous with respect to the seminorm $|\sigma^K(\cdot)|_K$ on $E$. It is elementary to check

$$\lim_{l\to\infty} P(\xi_l) \in |x|_K V_k \text{ and } \lim_{l\to\infty} Q(\xi_l) \in |\sigma^N_K(x)|_N U_n$$

which proves $(\widetilde{S}_3)$. As a next step we show that $(\widetilde{S}_3)$ implies that $F$ is a quojection whenever $E$ is not countably normed. Indeed, in this case we have

$$(\star) \quad \forall N \in \mathbb{N}\ \exists\, M \ge N\ \forall\, K \ge M \quad \ker(\sigma^M_K) \ne \ker(\sigma^N_K)$$

since otherwise it would be easy to construct a representing spectrum for $E$ with injective spectral maps. Let $n \le m \le k$ and $N$ be as in $(\widetilde{S}_3)$ and choose $M \in \mathbb{N}$ according to $(\star)$ and again $K \in \mathbb{N}$ and $S > 0$ as in $(\widetilde{S}_3)$. Now, $(\star)$ gives an $x \in E_K$ with $\sigma^M_K(x) \ne 0$ and $\sigma^N_K(x) = 0$. This implies $\varrho^n_m(V_m) \subseteq C\varrho^n_k(V_k)$ with $C = S|x|_K\left(|\sigma^M_K(x)|_M\right)^{-1}$ hence $\varrho^n_m(F_m) = \varrho^n_k(F_k)$. This yields that $F$ is quojection.

Now, we are ready to prove the proposition. Part 4. is simple because if $E$ is countably normed the inequalities in the definition of $(S^*_3)$ become trivial for $f \notin F^*_n$ if we choose the seminorms $|\cdot|_N$ is such a way that $\sigma^N_M$ are injective.

To prove 1. we first note that $(\widetilde{S}_3)$ implies $(S^\bullet_3)$. If $E$ is countably normed, 4. gives $(S^*_3)$. If $E$ is not countably normed, $(\widetilde{S}_3)$ implies that $F$ is quojection and this yields $(S^*_3)$ if we choose seminorms $\|\cdot\|_n$ on $F$ such that the canonical maps between the Hausdorff completions become surjective hence open. Then $\|f\|^*_m \le C' \cdot \|f\|^*_k$ for all $f \in F^*_m$ and some constant $C'$ depending only on $m$ and $k$ and the inequalities in $(S^*_3)$ are thus trivial.

For the second part we still have to show that $F$ is a quojection whenever $(E, F)$ satisfies $(S^*_3)$ for some non-countably normed space $E$. But it follows exactly as above that we then have

$$\forall n \in \mathbb{N}\ \exists\, m \ge n\ \forall\, k \ge m\ \exists\, \alpha > 0\ \forall\, f \in F^*_m \quad \|f\|^*_m \le \alpha \|f\|^*_k.$$

Hence there is a strictly increasing sequence $(n_j)_{j\in\mathbb{N}}$ of natural numbers such that $F^*_{n_j}$ is a topological (hence closed) subspace of $F^*_{n_{j+1}}$. Using the closed range theorem we can then deduce that $F$ is a quojection.

Finally, it is easily seen that for an $E$-acyclic quojection $F$ the projective spectrum $(L(E, F_n), R^n_m)$ is equivalent to a spectrum with surjective spectral maps hence $\operatorname{Proj}^1$ vanishes on this spectrum. Together with proposition 5.1.5 this proves 3. □

**Remark 5.2.3** The same argument as in the above proof shows that for a non countably normed space $E$ condition $(S_3^\bullet)$ implies that $F'$ is a strict (LB)-space, i.e. it has *some* representation $F' = \operatorname{ind} Z_n$ with Banach spaces $Z_n$ such that $Z_n$ is a topological subspace of $Z_{n+1}$. These spaces need not be the duals of $F_n$ and thus, it is not possible to apply the closed range theorem to deduce that $F$ is a quojection (one only gets that $F''$ is a quojection). In fact, according to a result of E. Behrends, S. Dierolf and P. Harmand [4] there exist Fréchet spaces $F$ which fail to be a quojection but yet having a strict (LB)-space as the dual (such spaces are called prequojections). It is easy to see that for a prequojection $F$ the pair $(\omega, F)$ satisfies $(S_3^\bullet)$ but not $(S_3^*)$ and we get from the first part of 5.2.2 $\operatorname{Ext}^1(\omega, F) \neq 0$. This clarifies a complaint of the reviewer of [29] in the Mathematical Reviews (MR97g:46095).

Let now $(Y, \|\cdot\|)$ be any Banach space and $p$ a strictly coarser norm on $Y$. Then $F = Y^{\mathbb{N}}$ is a quojection and we have $\operatorname{Ext}^1(\omega, F) = 0$. However, if we take

$$\|(y_l)_{l\in\mathbb{N}}\|_n = \sum_{l=1}^{n-1} \|y_l\| + p(y_n) \text{ and } |(x_l)_{l\in\mathbb{N}}|_N = \sum_{l=1}^{N} |x_l|$$

as fundamental sequences of seminorms on $F$ and $E = \omega$, respectively, the inequalities of condition $(S_3^*)$ do not hold for these seminorms although $(\omega, F)$ does satisfy condition $(S_3^*)$.

As a preparation for the main theorem of this section we now prove a variant of [63, lemma 3.3].

**Lemma 5.2.4** *Let $E$ be a non-normable Fréchet space and $F$ a Fréchet space such that $(E, F)$ satisfies $(S_3^\bullet)$. If $\mathscr{E} = (E_n, \sigma_M^N)$ and $\mathscr{F} = (F_n, \varrho_m^n)$ are representing spectra for $E$ and $F$ we have*

$$\forall\ n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\, N \in \mathbb{N}\ \forall\, M \geq N,\, \varepsilon > 0\ \exists\, K \geq M, S > 0$$

$$\forall x \in E_k,\, f \in F_n' \quad |\sigma_K^M(x)|_M \|f \circ \varrho_m^n\|_m^* \leq S|x|_K \|f \circ \varrho_k^n\|_k^* + \varepsilon |\varrho_K^N(x)|_N \|f\|_n^*.$$

**Proof.** We first show that the spectrum $\mathscr{F}$ satisfies

$$(Q) \quad \forall n \in \mathbb{N}\ \exists\, \widetilde{n} \geq n\ \forall\, k \geq \widetilde{n},\, \delta > 0\ \exists\, C > 0\ \forall\, f \in F_n'$$

$$\|f \circ \varrho_{\widetilde{n}}^n\|_{\widetilde{n}}^* \leq C \|f \circ \varrho_k^n\|_k^* + \delta \|f\|_n^*.$$

Indeed, for $n \in \mathbb{N}$ we take $\widetilde{n} = m$ from $(S_3^\bullet)$ and choose for $k \geq \widetilde{n}$ again $N \in \mathbb{N}$ from $(S_3^\bullet)$ and $M > N$ such that $|\sigma^N(\cdot)|_N$ and$|\sigma^M(\cdot)|_M$ are not equivalent on $E$. Once more we use $(S_3^\bullet)$ to get $K \geq M$ and $S > 0$. Given $\delta > 0$ there is $x \in E$ with

$$0 < |\sigma^N(x)|_N \leq \delta S^{-1} \left|\sigma^M(x)\right|_M. \text{ With } C = S\frac{|\sigma^K x|_K}{|\sigma^N x|_N} \text{ we obtain}$$

$$\|f \circ \varrho_{\widetilde{n}}^n\|_{\widetilde{n}}^* \leq C \|f \circ \varrho_k^n\|_k^* + \delta \|f\|_n^*.$$

Now we prove the assertion of the lemma. If $n \in \mathbb{N}$ is given we apply $(S_3^\bullet)$ to $\widetilde{n}$ and obtain $m \geq \widetilde{n}$. Given $k \geq m$ we get $N \in \mathbb{N}$. For $M \geq N$ we choose again $S > 0$ according to $(S_3^\bullet)$, and given $\varepsilon > 0$ we use $(Q)$ to find $C > 0$ according to $\delta = \varepsilon/S$. For $f \in F_n'$ and $x \in E_K$ we obtain

$$\begin{aligned}
&|\sigma_K^M(x)|_M \|f \circ \varrho_{\widetilde{n}}^n \circ \varrho_m^{\widetilde{n}}\|_m^* \\
&\leq S|x|_K \|f \circ \varrho_{\widetilde{n}}^n \circ \varrho_k^{\widetilde{n}}\|_k^* + S|\varrho_K^N(x)|_N \|f \circ \varrho_{\widetilde{n}}^n\|_{\widetilde{n}}^* \\
&\leq S|x|_K \|f \circ \varrho_k^n\|_k^* + SC|\sigma_K^N(x)|_N \|f \circ \varrho_k^n\|_k^* + \varepsilon|\sigma_K^N(x)|_N \|f\|_n^* \\
&\leq \widetilde{S}|x|_K \|f \circ \varrho_k^n\|_k^* + \varepsilon|\sigma_K^N(x)|_N \|f\|_n^*,
\end{aligned}$$

where $\widetilde{S} = S(1+C)|\sigma_K^N|$ and $|\sigma_K^N|$ is the norm of the operator $\sigma_K^N$. □

**Theorem 5.2.5** *Let $E$ and $F$ be Fréchet spaces such that $E$ is nuclear. Then* $\mathrm{Ext}^1(E,F) = 0$ *if and only if $(E,F)$ satisfies $(S_3^*)$.*

**Proof.** If $E$ is not countably normed the result is covered by proposition 5.2.2, and it is obvious if $E$ is normable (hence finite dimensional). We first show that the condition in the previous lemma implies $(\overline{S}_3)$:

$$\forall\ n \in \mathbb{N}\ \exists\ m \geq n\ \forall\ k \geq m\ \exists\ N \in \mathbb{N}\ \forall\ M \geq N,\ \varepsilon > 0\ \exists\ K \geq M, S > 0$$

$$\forall x \in E_K \quad |\sigma_K^M(x)|_M \varrho_m^n(V_m) \subseteq S|x|_K \varrho_k^n(V_k) + \varepsilon|\sigma_K^N(x)|_N V_n,$$

where $(E_N, \sigma_M^N)$ and $(F_n, \varrho_m^n)$ are representing spectra for $E$ and $F$, respectively, and $V_n$ are the closed unit balls of $F_n$. Indeed, with the same quantifiers as above, lemma 5.2.4 gives

$$a\|f \circ \varrho^n\|_m^* \leq b\|f \circ \varrho_k^n\|_k^* + c\|f\|_n^*$$

for all $f \in F_n'$ where $a = |\sigma_K^M(x)|_M$, $b = S|x|_K$, $c = \varepsilon|\sigma_K^N(x)|_N$ are either all 0 or all positive since $E$ is countably normed and thus, $\sigma_K^N$ can be assumed to be injective. This yields (with $g^{-t} := (g^t)^{-1}$)

$$\frac{1}{c}V_n^\circ \cap \frac{1}{b}(\varrho_k^n)^{-t}(V_k^\circ) \subseteq \frac{2}{a}(\varrho_m^n)^{-t}(V_m^\circ)$$

and since $((\varrho_m^n)^{-t}(V_m^\circ))^\circ = \varrho_m^n(V_m)^{\circ\circ}$ the theorem of bipolars gives

$$\frac{a}{2}\varrho_m^n(V_m) \subseteq \overline{b\varrho_k^n(V_k)^{\circ\circ} + cV_n^{\circ\circ}}^{F_n} \subseteq b\varrho_k^n(V_k) + 2cV_n.$$

This implies $(\overline{S}_3)$ (with $\widetilde{S} = 4S$).

By passing to an equivalent representing spectrum of $E$ we may assume that all $E_N$ are Hilbert spaces (with scalar products $\langle\cdot,\cdot\rangle_N$) such that $\sigma^N(E)$ is dense in $E_N$, and that the spectral maps are Hilbert-Schmidt operators. As before, we denote

$$B_{n,N} = \left\{T \in L(E, F_n) : \sup\{\|Tx\|_n : |\sigma^N(x)|_N \leq 1\}\right\}.$$

We will show that with the same quantifiers as in $(\overline{S}_3)$ we have

$$R_m^n\left(B_{m,M}\right) \subseteq \widetilde{S} R_k^n\left(B_{k,K+1}\right)+\widetilde{\varepsilon} B_{n,N+1}$$

with $\widetilde{S} = S\nu_2(\sigma_{K+1}^K)$ and $\widetilde{\varepsilon} = \varepsilon\nu_2(\sigma_{N+1}^N)$ where $\nu_2(\cdot)$ denotes the Hilbert-Schmidt norm.

We choose a complete orthonormal system $(e_i)_{i\in I}$ in $E_{K+1}$ and an orthonormal system $(f_i)_{i\in I}$ in $E_{N+1}$ such that

$$\sigma_{K+1}^{N+1} = \sum_{i\in I} a_i \langle \cdot, e_i\rangle_{k+1} f_i$$

with positive constants $a_i$. Let $T \in B_{m,M}$ be given and let $\widetilde{T} : E_M \longrightarrow F_m$ be its unique factorization (i.e. $T = \widetilde{T} \circ \sigma^M$). Since $\widetilde{T}$ is a contraction we have

$$u_i := \widetilde{T}(\sigma_{K+1}^M(e_i)) \in |\sigma_{K+1}^M(e_i)|_M\, V_m$$

and $(\overline{S}_3)$ implies the existence of

$$v_i \in S|\sigma_{K+1}^K(e_i)|_K\, V_k \text{ and } w_i \in \varepsilon|\sigma_{K+1}^N(e_i)|_N V_n$$

with $\varrho_m^n(u_i) = \varrho_k^n(v_i) + w_i$. We now define operators

$$V : E_{K+1} \longrightarrow F_k \text{ by } V(x) = \sum_{i\in I} \langle x, e_i\rangle_{K+1} v_i, \text{ and}$$
$$W : E_{N+1} \longrightarrow F_n \text{ by } W(x) = \sum_{i\in I} \frac{1}{a_i}\langle x, f_i\rangle_{N+1} w_i.$$

These are indeed continuous linear operators since for $x \in E_{K+1}$

$$\begin{aligned}
&\sum_{i\in I} |\langle x, e_i\rangle_{K+1}|\|v_i\|_k \le S\sum_{i\in I} |\langle x, e_i\rangle_{K+1}||\sigma_{K+1}^K(e_i)|_K \\
\le{}& S\left(\sum_{i\in I} |\langle x, e_i\rangle_{K+1}|^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} |\sigma_{K+1}^K(e_i)|_K^2\right)^{\frac{1}{2}} = S|x|_{K+1}\nu_2\left(\sigma_{K+1}^K\right),
\end{aligned}$$

and since $\sigma_{K+1}^{N+1}(e_i) = a_i f_i$ we have for $x \in E_{N+1}$

$$\begin{aligned}
&\sum_{i\in I} \frac{1}{a_i}|\langle x, f_i\rangle_{N+1}|\|w_i\|_n \le \varepsilon \sum_{i\in I} \frac{1}{a_i}|\langle x, f_i\rangle_{N+1}||\sigma_{K+1}^N(e_i)|_N \\
={}& \varepsilon\sum_{i\in I} |\langle x, f_i\rangle_{N+1}||\sigma_{N+1}^N(f_i)|_N \le \varepsilon|x|_{N+1}\nu_2(\sigma_{N+1}^N).
\end{aligned}$$

Given $x = \sum_{i\in I} \langle x, e_i\rangle_{K+1} e_i \in E_{K+1}$ we have

$$W\left(\sigma_{K+1}^{N+1}(x)\right)=\sum_{i\in I}\langle x,e_i\rangle_{K+1}W(a_if_i)=\sum_{i\in I}\langle x,e_i\rangle_{K+1}w_i \text{ and}$$

$$V(x)=\sum_{i\in I}\langle x,e_i\rangle_{K+1}V(e_i)=\sum_{i\in I}\langle x,e_i\rangle_{K+1}v_i,$$

which implies $\widetilde{T}\circ\sigma_{K+1}^M=V+W\circ\sigma_{K+1}^N$.

Defining $V_0=V\circ\sigma^{K+1}$ and $W_0=W\circ\sigma^N$ we obtain the desired decomposition $\varrho_m^n\circ T=\varrho_k^n\circ V_0+W_0$ with $V_0\in\widetilde{S}B_{k,K+1}$ and $W_0\in\widetilde{\varepsilon}B_{n,N+1}$. Since every bounded set in $L_b(E,F_n)$ is contained in some $B_{n,N}$ which are Banach discs we can apply theorem 3.2.14 and obtain $\mathrm{Proj}^1(L(E,F_n),R_m^n)=0$ and thus $\mathrm{Ext}^1(E,F)=0$ by proposition 5.1.5. □

Let us remark that metrizability of $E$ in the theorem above is used only for the necessity of $(S_3^*)$. The application of 3.2.14 only requires the sets $B_{n,U}:=\{T\in L(E,F_n):\sup\{\|Tx\|_n : x\in U\}\le 1\}$ to be Banach discs which is implied by the completeness of $F_n$. Thus, the proof of 5.2.5 shows the following result:

**Theorem 5.2.6** *Let $E$ be a nuclear locally convex space and $F$ a Fréchet space such that*

$$\forall\ n\in\mathbb{N}\ \exists\ m\ge n\ \forall\ k\ge m\ \exists\ U\in\mathscr{U}_0(E)\ \forall\ V\in\mathscr{U}_0(E)\ \exists\ W\in\mathscr{U}_0(E)$$

$$\forall\ x\in E_W\quad |\sigma_W^V(x)|_V\,\varrho_m^n(V_m)\subseteq|x|_W\,\varrho_k^n(V_k)+|\sigma_W^U(x)|_U V_n$$

*where $(E_W,|\cdot|_W)$ is the Hausdorff completion of $(E,p_W)$, $\sigma_W^V$ are the canonical maps, and $\mathscr{F}=(F_n,\varrho_m^n)$ is a representing spectrum for $F$ consisting of Banach spaces $F_n$ with unit balls $V_n$. Then $\mathrm{Ext}^k(E,F)=0$ for all $k\ge 1$.*

**Remark 5.2.7** Theorem 5.2.5 also holds for pairs $(E,F)$ of proper (i.e. non normable) Fréchet spaces where either $E=\lambda(A)$ (a Köthe space of order 1) or $F=\lambda^\infty(B)$ (a Köthe space of order $\infty$) or $F$ is nuclear. The proofs are very similar and differ from the one for 5.2.5 only in the decomposition of $T\in B_{m,M}$ (which is simpler in the case of Köthe spaces and a bit harder in the case where only $F$ is nuclear). The details can be found in [63].

The results in 3.3.4, 3.3.6, and 5.1.5 relate $\mathrm{Ext}^1(E,F)=0$ to barrelledness properties of the space $L_b(E,F)$. For instance, if $E$ is nuclear and countably normed then $L_b(E,F)$ is barrelled or ultrabornological if and only if $(E,F)$ satisfies $(S_3^*)$. Similar results had been obtained by Grothendieck [32], Krone and Vogt [40] and Vogt [61].

Let us recall that a Fréchet space $E$ with fundamental sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of the continuous seminorms satisfies (or belongs to the class) $(DN)$ if

$$\exists\ N\in\mathbb{N}\ \forall\ M\ge N\ \exists\ K\ge M,\ C>0\ \forall\ x\in E\quad \|x\|_M^2\le C\|x\|_K\|x\|_N,$$

and it satisfies (or belongs to the class) $(\Omega)$ if

$$\forall\, n\in\mathbb{N}\ \exists\, m\ge n\ \forall\, k\ge m\ \exists\, D>0,\,\alpha>0\ \forall\, f\in E'$$

$$(\|f\|_m^*)^{1+\alpha} \leq D\|f\|_k^* \, (\|f\|_n^*)^\alpha \,.$$

These conditions play a prominent role in the structure theory of Fréchet spaces. It is easy to check that the space

$$s = \left\{(x_j)_{j\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \|x\|_n := \sup_{j\in\mathbb{N}} |j^n x_j| < \infty \text{ for all } n \in \mathbb{N}\right\}$$

satisfies $(DN)$ as well as $(\Omega)$, that $(DN)$ is stable with respect to subspaces, and that $(\Omega)$ is stable with respect to quotients. Theorem 5.2.5 covers the classical splitting result for Fréchet spaces with (DN) and $(\Omega)$ [59, 67].

**Corollary 5.2.8** *Let $E$ be a nuclear Fréchet space with $(DN)$ and $F$ a Fréchet space with $(\Omega)$. Then* $\mathrm{Ext}^1(E,F) = 0$.

**Proof.** Let $(|\cdot|_N)_N$ and $\|\cdot\|_n$ be fundamental sequences of seminorms for $E$ and $F$ respectively. We will show condition $(S_3^*)$ for $(E,F)$. For $n \leq m \leq k$, $\alpha$ and $D$ as in $(\Omega)$ we take $N \in \mathbb{N}$ from the definition of $(DN)$. Iterating condition $(DN)$ shows $|x|_M^{2^p} \leq \widetilde{C}|x|_K|x|_N^{2^p-1}$ for some $K \geq M$ and $\widetilde{C} > 0$. Using this it is easy to find $K \geq M$ and $C > 0$ such that $|x|_M^{1+\alpha} \leq C|x|_K|x|_N^\alpha$ for all $x \in E$. For $x \in E$, $f \in F'$ we then have

$$|x|_M\|f\|_m^* \leq (CD+1)\left(|x|_K\|f\|_k^* + |x|_N\|f\|_n^*\right).$$

Indeed, this is obvious if $|x|_M\|f\|_m^* \leq |x|_N\|f\|_n$ and otherwise we multiply the $(\Omega)$ and modified (DN) inequalities to obtain

$$(|x|_M\|f\|_m^*)^{1+\alpha} \leq CD|x|_K\|f\|_k^*\,(|x|_N\|f\|_n^*)^\alpha \leq CD|x|_K\|f\|_k^*\,(|x|_M\|f\|_m^*)^\alpha$$

which shows $|x|_m\|f\|_m^* \leq CD|x|_K\|f\|_k^*$. By continuity, we get the same inequality for every $x \in E_K$, the completion of the normed space $(E, |\cdot|_K)$. Now the result follows from theorem 5.2.5. □

There are several further splitting results for Fréchet spaces. For instance, corollary 5.2.8 holds without any nuclearity assumption for hilbertizable Fréchet spaces (i.e. projective limits of Hilbert spaces), this is contained in the book of Meise and Vogt [45, §30], or more generally, if $E$ is (a projective limit of Banach spaces) of type 2 and $F$ is the bidual of a projective limit of Banach spaces with duals of type 2, which is due to Defant, Domański and Mastyło [22]. Moreover, condition $(S_3^*)$ allows further evaluation, if $E$ or $F$ are power series spaces, see [63].

For later use and for the sake of comparison we will deduce from 5.2.5 some results about the structure of nuclear Fréchet spaces. The first result (due to Vogt and Wagner [67]) follows from corollary 5.2.8 since $(DN)$ is stable with respect to subspaces and $(\Omega)$ is stable with respect to quotients.

**Corollary 5.2.9** *If $E$ is isomorphic to a subspace of $s$ and $F$ is isomorphic to a quotient of $s$ then* $\mathrm{Ext}^1(E,F) = 0$.

Together with the Kōmura–Kōmura theorem [38] (every nuclear Fréchet space is a subspace of $s^{\mathbb{N}}$) and Pełczyński's decomposition trick this yields the next result taken from [23] and [67].

**Corollary 5.2.10** *Let $E$ and $F$ be Fréchet spaces.*

1. *If $E$ is isomorphic to a subspace of $s$ then there exists an exact sequence*

$$0 \longrightarrow E \longrightarrow s \longrightarrow s \longrightarrow 0.$$

2. *If $F$ is isomorphic to a quotient of $s$ then there exists an exact sequence*

$$0 \longrightarrow s \longrightarrow s \longrightarrow F \longrightarrow 0.$$

3. *If $E$ is isomorphic to a subspace of $s$ and to a quotient of $s$ then $E$ is isomorphic to a complemented subspace of $s$.*

**Proof.** Since this result is essentially contained in §31 of the book [45] we only sketch the proof for the reader's convenience. One first uses Borel's theorem [45, 26.29] to find an exact sequence

$$0 \longrightarrow s \longrightarrow s \longrightarrow \mathbb{K}^{\mathbb{N}} \longrightarrow 0$$

and this yields an exact sequence

$$0 \longrightarrow s \longrightarrow s \longrightarrow s^{\mathbb{N}} \longrightarrow 0$$

by considering spaces $s(X)$ of vector valued rapidly decreasing sequences [45, 31.3]. Using the Kōmura–Kōmura theorem [45, 29.9] one gets that for every nuclear Fréchet space $G$ there are a subspace $\widetilde{G}$ of $s$ and an exact sequence

$$0 \longrightarrow s \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 0.$$

If $E$ is a subspace of $s$ and $G = s/E$, the pull back procedure together with the remark after after definition 5.1.2 gives the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & s & \longrightarrow & \widetilde{G} & \longrightarrow & s/E & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & s & \longrightarrow & H & \longrightarrow & s & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & E & = & E & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The second row splits by 5.2.9, hence $H \cong s \times s \cong s$. Form this we first deduce part 3. If $E$ is also isomorphic to a quotient of $s$ the first column splits by 5.2.9, hence $E$ is a complemented subspace of $H \cong s$.

Going back to the case where $E$ is only a subspace of $s$ we deduce that $\widetilde{G}$ is isomorphic to a subspace and to a quotient of $s$ hence $\widetilde{G}$ is isomorphic to a complemented subspace of $s$ by 3. Forming the product of the first row with the trivial sequence

$$0 \longrightarrow s \xrightarrow{id} s \longrightarrow 0 \longrightarrow 0$$

we get an exact sequence

$$0 \longrightarrow s \longrightarrow \widetilde{G} \times s \longrightarrow s/E \longrightarrow 0.$$

If $K$ is a complement of $\widetilde{G}$ in $s$ we obtain with Pełczyński's decomposition method

$$\begin{aligned} \widetilde{G} \times s &\cong \widetilde{G} \times s(s) \cong \widetilde{G} \times s(\widetilde{G} \times K) \cong \widetilde{G} \times s(\widetilde{G}) \times s(K) \\ &\cong s(\widetilde{G}) \times s(K) \cong s(\widetilde{G} \times K) \cong s(s) \cong s. \end{aligned}$$

This proves the second part, and the first is obtained in the same way if we form the product of the first column of the diagram with

$$0 \longrightarrow 0 \longrightarrow s \xrightarrow{id} s \longrightarrow 0$$

□

Note that the full strength of 5.2.9 was only used to prove the third part. Parts 1. and 2. follow from 3. and $\mathrm{Ext}^1(s,s) = 0$. (However, since $\mathrm{Ext}^2(E,F) = 0$ holds for all nuclear Fréchet spaces by proposition 5.1.5 one can deduce corollary 5.2.9 from $\mathrm{Ext}^1(s,s) = 0$ using the long cohomology sequences from theorem 5.1.1.)

In the next section we will prove a complete analogon of corollary 5.2.9 in the category of (PLS)-spaces, i.e. countable projective limits of (LS)- spaces, where the space $\mathscr{D}' \cong (s')^{\mathbb{N}}$ plays the role of $s$. The proof is just the other way round. The analogon of the third part of 5.2.10 is a theorem of Domański and Vogt [24] from which one can deduce the second, and this finally yields the desired splitting theorem for subspaces and quotients of $\mathscr{D}'$.

## 5.3 Splitting in the category of (PLS)-spaces

In this section we investigate the splitting of (topologically) exact sequences

$$0 \longrightarrow F \longrightarrow G \longrightarrow E \longrightarrow 0$$

where $E, F, G$ are $(PLS)$-spaces, i.e. countable projective limits of $(LS)$-spaces. The splitting of all such sequences is denoted by $\mathrm{Ext}^1_{PLS}(E,F) = 0$. Note the difference to $\mathrm{Ext}^1(E,F) = 0$ which is equivalent to the splitting of all these exact sequences where $G$ is any locally convex space.

We start with some properties of the category of $(PLS)$-spaces as in [25]. We use the fact that the class of $(LS)$-spaces is stable with respect to closed subspaces, quotients, and finite products, which implies that the category of $(LS)$-spaces and continuous linear maps is quasi-abelian (the pull back is a closed subspace of a product and the push out is a quotient of a product, as we have seen after 5.1.2). An algebraically short exact sequence of $(LS)$-spaces with continuous maps is automatically topologically exact (which follows e.g. from the Köthe–Grothendieck open mapping theorem), and it splits if and only if the dual exact sequence of Fréchet spaces splits (the transposed of a right inverse is a projection). The duality between complexes of $(LS)$-spaces and complexes of Fréchet–Schwartz spaces is thus "perfect", more information about duals of exact sequences is contained in chapter 7.

**Proposition 5.3.1** *Closed subspaces and complete quotients of $(PLS)$-spaces are again $(PLS)$-spaces, and the category of $(PLS)$-spaces is quasi-abelian.*

**Proof.** Let $X$ be a $(PLS)$-space and $\mathscr{X} = (X_n, \varrho^n_m)$ a strongly reduced spectrum of $(LS)$-spaces with $\mathrm{Proj}\, \mathscr{X} \cong X$. If $E$ is a closed subspace of $X$, we set $E_n = \overline{\varrho^n(E)}^{X_n}$ and $\sigma^n_m = \varrho^n_m|_{E_m}$. Then $\mathscr{E} = (E_n, \sigma^n_m)$ is a spectrum of $(LS)$-spaces with $\mathrm{Proj}\,\mathscr{E} \cong E$. Moreover, $\mathscr{F} = (X_n/E_n, \tau^n_m)$, where $\tau^n_m : X_m/E_m \longrightarrow X_n/E_n$ is the mapping induced by $\varrho^n_m$, is a spectrum of $(LS)$-spaces, such that $X/E \longrightarrow \mathrm{Proj}\,\mathscr{F}$, $x + E \longmapsto (\varrho^n(x) + E_n)_{n\in\mathbb{N}}$ is an isomorphism onto a dense subspace of $\mathrm{Proj}\,\mathscr{F}$. If $X/E$ is complete, we obtain $X/E \cong \mathrm{Proj}\,\mathscr{F}$. As we have said above, the pull back (in the category of locally convex spaces) is a closed subspace of a product of two $(PLS)$-spaces, hence itself a $(PLS)$-space. If

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is an exact sequence of $(PLS)$-space, and $T : X \longrightarrow E$ is a morphism we obtain the push out in the category of locally convex spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & Z & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

where $F$ is a quotient of the $(PLS)$-space $E \times Y$. Since $E$ and $Z$ are complete and completeness is a three space property (this follows e.g. from the considerations in section 4.2), $F$ is complete and therefore also a PLS-space. □

Let us note that proposition 5.1.3 applies to the category of $(PLS)$-spaces, i.e. the splitting condition we took as the definition of $\mathrm{Ext}^1_{PLS}(E,F)$ is equivalent to the lifting and extension properties in 5.1.3. This easily implies that the class of $(PLS)$-spaces $F$ with $\mathrm{Ext}^1_{PLS}(E,F) = 0$ is stable with respect to countable products.

We now show that short exact sequences of $(PLS)$-spaces induce exact sequences of corresponding spectra.

**Proposition 5.3.2** *Let* $(\star)\quad 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *be an exact sequence of* $(PLS)$*-spaces. Then there are strongly reduced spectra* $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ *of* $(LS)$*-spaces and a sequence*

$$0 \longrightarrow \mathscr{X} \longrightarrow \mathscr{Y} \longrightarrow \mathscr{Z} \longrightarrow 0$$

*which is exact in the category of locally convex spectra such that* $(\star)$ *is the projective limit of this sequence. Moreover, if* $\widetilde{\mathscr{X}}, \widetilde{\mathscr{Y}}, \widetilde{\mathscr{Z}}$ *are strongly reduced spectra of* $(LS)$*-spaces having* $X, Y,$ *and* $Z$*, respectively, as projective limits we can take either* $\mathscr{Y} = \widetilde{\mathscr{Y}}$ *or* $\mathscr{X}$ *and* $\mathscr{Z}$ *as subsequences of* $\widetilde{\mathscr{X}}$ *and* $\widetilde{\mathscr{Z}}$*.*

**Proof.** If $\mathscr{Y} = (Y_n, \sigma^n_m)$ is a strongly reduced spectrum with $\mathrm{Proj}\,\mathscr{Y} \cong Y$ we can take $\mathscr{X}$ and $\mathscr{Z}$ as the spectra consisting of $X_n = \overline{\mathrm{im}(\sigma^n)}$ and $Y_n/X_n$. Since strongly reduced spectra of $(LS)$-spaces are equivalent by proposition 3.3.8, we have factorizations

$$\begin{array}{ccccccccc} & & \widetilde{X}_m & & & & & & \\ & & \uparrow & & & & & & \\ 0 & \longrightarrow & X_n & \longrightarrow & Y_n & \longrightarrow & Y_n/X_n & \longrightarrow & 0 \\ & & & & & & \uparrow & & \\ & & & & & & \widetilde{Z}_k & & \end{array}$$

As we have noted above, the category of $(LS)$-spaces is quasi-abelian and we can thus use pull back and push out to obtain

$$0 \longrightarrow \mathscr{X}' \longrightarrow \mathscr{Y}' \longrightarrow \mathscr{Z}' \longrightarrow 0$$

where $\mathscr{X}'$ and $\mathscr{Z}'$ are subsequences of $\widetilde{\mathscr{X}}$ and $\widetilde{\mathscr{Z}}$, respectively. □

The aim of this section is to show that the $(PLS)$-space $(s')^{\mathbb{N}}$ plays the same role for the splitting in the category of $(PLS)$-spaces as $s$ does in the category of Fréchet spaces. This is important for distribution theory, because for any open subset $\Omega \subseteq \mathbb{R}^N$ the space $\mathscr{D}'(\Omega)$ of distributions (in the sense of

L. Schwartz) is isomorphic to $(s')^{\mathbb{N}}$. We will not need this result of Valdivia [57] and (independently) Vogt [60]. Nevertheless we will use the abbreviation $\mathscr{D}' := (s')^{\mathbb{N}}$. The first splitting result is simple:

**Theorem 5.3.3** *If $E$ is isomorphic to a subspace of $\mathscr{D}'$ and $F_n$ are isomorphic to quotients of $s'$ then* $\operatorname{Ext}^1_{PLS}(E, \prod_{n\in\mathbb{N}} F_n) = 0$.

**Proof.** By what we have said after 5.3.1 we only have to show $\operatorname{Ext}^1_{PLS}(E,F) = 0$ for a quotient $F$ of $s'$. There is a strongly reduced spectrum $\mathscr{E} = (E_n, \varrho^n_m)$ consisting of subspaces of $s'$ with $\operatorname{Proj}\mathscr{E} = E$. If now

$$0 \longrightarrow F \xrightarrow{\ i\ } G \xrightarrow{\ q\ } E \longrightarrow 0$$

is an exact sequence of $(PLS)$-spaces and $\mathscr{F} = (F, id)$ is the constant spectrum we apply 5.3.2 to get a spectrum $\mathscr{G} = (G_n, \sigma^n_m)$ of LS-spaces such that (after renumbering the spaces $E_n$) we have commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{\ i\ } & G & \xrightarrow{\ q\ } & E & \longrightarrow & 0 \\ & & {\scriptstyle id}\downarrow & & {\scriptstyle \tau^n}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & F & \xrightarrow{\ i_n\ } & G_n & \xrightarrow{\ q_n\ } & E_n & \longrightarrow & 0 \end{array}$$

Considering the dual of the second row (which is exact since the spaces involved are $(LS)$) and using $\operatorname{Ext}^1(F', E_n') = 0$ we see that the second row splits. If $\pi_n$ is a projection $G_n \longrightarrow F$ we obtain a projection $\pi_n \circ \tau^n$ from $G$ onto $F$, hence the first row splits, too. □

We will now present one of the main results of Domański and Vogt from [24]. We first give a slightly improved version of [24, lemma 6]. A $(PLS)$-space $X$ is called strict if there is a spectrum $\mathscr{X} = (X_n, \varrho^n_m)$ of $(LS)$-spaces with $\operatorname{Proj}\mathscr{X} \cong X$ and

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m \quad \varrho^n_m(X_m) = \varrho^n_k(X_k).$$

Applying theorem 3.2.5 to the discrete topologies yields that we then even have $\varrho^n_m(X_m) \subseteq \varrho^n(\operatorname{Proj}\mathscr{X})$.

**Lemma 5.3.4** *Let $0 \longrightarrow X \xrightarrow{\ i\ } Y \xrightarrow{\ q\ } Z \longrightarrow 0$ be a short exact sequence of $(PLS)$-spaces such that $X$ is strict. Let $\mathscr{Y} = (Y_n, \sigma^n_m)$ and $\mathscr{Z} = (Z_n, \tau^n_m)$ be strongly reduced spectra of $(LS)$-spaces representing $Y$ and $Z$. Then*

$$\forall\, n \in \mathbb{N}\ \exists\, m \in \mathbb{N}\ \forall\, B \in \mathscr{B}(Z)\ \exists\, D \in \mathscr{B}(Y)$$

$$B \cap \ker(\tau^m) \subseteq q\left(D \cap \ker(\sigma^n)\right).$$

**Proof.** We first note that the required property is independent under passing to equivalent spectra. Using lemma 5.3.2 we get commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\ i\ } & Y & \xrightarrow{\ q\ } & Z & \longrightarrow & 0 \\ & & {\scriptstyle \varrho^n}\downarrow & & {\scriptstyle \sigma^n}\downarrow & & {\scriptstyle \tau^n}\downarrow & & \\ 0 & \longrightarrow & X_n & \xrightarrow{\ i_n\ } & Y_n & \xrightarrow{\ q_n\ } & Z_n & \longrightarrow & 0. \end{array}$$

Since a quotient map between $(LS)$-spaces lifts bounded sets, theorems 3.3.11 and 3.3.14 imply that $q$ lifts bounded sets. Given $n \in \mathbb{N}$ we choose $m \geq n$ with $\varrho^n_m(X_m) \subseteq \varrho^n(X)$. If now $B$ is bounded in $Z$ we take $K \in \mathscr{B}(Y)$ with $B \subseteq q(K)$. Then $L := (i_m)^{-1}(\sigma^m(K))$ is bounded in $X_m$ and using that (for every index set $I$) the spectrum consisting of the spaces $\ell^\infty_I(X_n)$ is also strict we find a bounded set $M \subseteq X$ with $\varrho^n_m(L) \subseteq \varrho^n(M)$. Given $z \in B \cap \ker(\tau^m)$ we choose $y \in K$ with $z = q(y)$. Then $q_m(\sigma^m(y)) = \tau^m(z) = 0$, hence there is $x_m \in L$ with $i_m(x_m) = \sigma^m(y)$. If $x \in M$ satisfies $\varrho^n_m(x_m) = \varrho^n(x)$ we obtain

$$q(y - i(x)) = z,\ y - i(x) \in D := K - i(M) \in \mathscr{B}(Y)$$

$$\text{and } \sigma^n(y - i(x)) = \sigma^n_m(\sigma^m(y) - i_m(x)) = 0.$$

□

We will use below that the lemma is equivalent to the existence of a non-decreasing sequence $(n_m)_{m \geq m_0}$ tending to infinity such that

$$\forall m \geq m_0,\ B \in \mathscr{B}(Z)\ \exists\ D \in \mathscr{B}(Y) \quad B \cap \ker(\tau^m) \subseteq q\left(D \cap \ker(\sigma^{n_m})\right).$$

**Theorem 5.3.5** *Let $F$ be a $(PLS)$-space which is isomorphic to a subspace of $\mathscr{D}'$ and to a quotient of $\mathscr{D}'$. Then $F$ is isomorphic to a complemented subspace of $\mathscr{D}'$.*

**Proof.** We follow essentially the arguments of [24].

1. There is a strongly reduced spectrum $\mathscr{F} = (F_n, \varrho^n_m)$ with $F_n \cong s'$ and $\operatorname{Proj} \mathscr{F} \cong F$.

Indeed, there are strongly reduced spectra $\mathscr{X} = (X_n, \sigma^n_m)$ and $\mathscr{Y} = (Y_n, \tau^n_m)$ with $X_n$ isomorphic to subspaces of $s'$, $Y_n$ isomorphic to quotients of $s'$ and $\operatorname{Proj} \mathscr{X} \cong \operatorname{Proj} \mathscr{Y} \cong F$. Since $X_n'$ is isomorphic to a quotient of $s$, corollary 5.2.10 provides a short exact sequence

$$0 \longrightarrow s \longrightarrow s \longrightarrow X_n' \longrightarrow 0.$$

Since by 3.3.8 strongly reduced spectra of $(LB)$-spaces with the same limits are equivalent, we may assume that $\tau^n_{n+1}$ factorizes as $\tau^n_{n+1} = R_n \circ S_n$ with $R_n : X_n \longrightarrow Y_n$. Now $\operatorname{Ext}^1(Y_n', s)$ yields that $R_n^t : Y_n' \longrightarrow X_n'$ factorizes through $s$ which implies that $R_n$ (hence also $\tau^n_{n+1}$) factorizes through $s'$.

2. We now show that there are a strict $(PLS)$-space $G$ and an exact sequence $0 \longrightarrow G \longrightarrow \mathscr{D}' \longrightarrow F \longrightarrow 0$.

Since $F$ is a quotient of $\mathscr{D}'$ there is a short exact sequence

$$(*) \quad 0 \longrightarrow X \longrightarrow \mathscr{D}' \longrightarrow F \longrightarrow 0.$$

Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a strongly reduced spectrum such that all $X_n$ are isomorphic to subspaces of $s'$ and $\operatorname{Proj} \mathscr{X} \cong X$. Since every strongly reduced spectrum $\mathscr{Y}$ with $\operatorname{Proj} \mathscr{Y} \cong \mathscr{D}'$ satisfies $\operatorname{Proj}^1 \mathscr{Y} = 0$ and $(*)$ is the limit of an exact sequence of spectra, we obtain $\operatorname{Proj}^1 \mathscr{X} = 0$. By theorem 3.2.9 there are Banach discs $B_n \subseteq X_n$ such that (after omitting some of the steps $X_n$)

$$\varrho_{n+1}^n(X_{n+1}) \subseteq \varrho_k^n(X_k) + B_n \text{ for every } k \geq n+1.$$

$X_n$ is a quotient of $\bigoplus_{m \in \mathbb{N}} s'$, because $X_n'$ is a subspace on $s^{\mathbb{N}}$ by the Kōmura–Kōmura theorem. Since quotient maps between $(LS)$-spaces lift bounded sets and each bounded set in $\bigoplus_{n \in \mathbb{N}} s'$ is contained in a finite sum $\bigoplus_{m=1}^{M} s' \cong s'$, there are continuous linear maps $T_n : s' \longrightarrow X_n$ with $B_n \subseteq T_n(s')$.

Now we set $G_n = X_n \times (s')^n$ and define $\gamma_{n+1}^n : G_{n+1} \longrightarrow G_n$ by

$$(x, y_1, \ldots, y_n) \longmapsto (\varrho_{n+1}^n(x) + T_n(y_1), y_2, \ldots, y_n).$$

Setting $\gamma_m^n = \gamma_{n+1}^n \ldots \circ \gamma_m^{m-1}$ we obtain a spectrum $\mathscr{G} = (G_n, \gamma_m^n)$ of $(LS)$-spaces, and it is easily seen that $\gamma_{n+1}^n(G_{n+1}) \subseteq \gamma_{n+2}^n(G_{n+1})$ holds. If $f_n : X_n \longrightarrow G_n$ is the canonical embedding and $g_n : G_n \longrightarrow (s')^n$ is the canonical projection we obtain an exact sequence of projective spectra and the limit is the exact sequence

$$0 \longrightarrow X \longrightarrow G \longrightarrow \mathscr{D}' \longrightarrow 0,$$

where $G = \operatorname{Proj} \mathscr{G}$ is a strict $(PLS)$-space. Applying the push out construction we get a $(PLS)$-space $Y$ and a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathscr{D}' & = & \mathscr{D}' & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & G & \longrightarrow & Y & \longrightarrow & F & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & \mathscr{D}' & \longrightarrow & F & \longrightarrow & 0. \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The second column splits by theorem 5.3.3, hence $Y \cong \mathscr{D}' \times \mathscr{D}' \cong \mathscr{D}'$ and the upper row gives the desired exact sequence

$$0 \longrightarrow G \longrightarrow \mathscr{D}' \longrightarrow F \longrightarrow 0.$$

3. Let now $\mathscr{F} = (F_n, \varrho_m^n)$ be a spectrum as constructed in 1. Since $F$ is a strict $(PLS)$-space (as a quotients of $\mathscr{D}'$) $\operatorname{Proj}^1 \mathscr{F} = 0$ holds and the canonical sequence

$$0 \longrightarrow F \xrightarrow{i} \prod_{n\in\mathbb{N}} F_n \xrightarrow{\Psi} \prod_{n\in\mathbb{N}} F_n \longrightarrow 0$$

is topologically exact. We now apply (the equivalent formulation of) lemma 5.3.4 to obtain $m_0 \in \mathbb{N}$ and a non-decreasing sequence $(n_m)_{m\geq m_0}$ tending to infinity such that for every $m \geq m_0$ every bounded set of $\prod_{l<m} \{0\} \times \prod_{l\geq m} F_l$ is contained in the $\Psi$-image of a bounded subset of

$$\prod_{l<n_m} \{0\} \times \prod_{l\geq n_m} F_l =: A_{n_m}.$$

Let $j_m : F_m \longrightarrow \prod_{l\in\mathbb{N}} F_l$ be the canonical inclusion and

$$H_{n_m} = A_{n_m} \cap \Psi^{-1}(j_m(F_m)).$$

Denoting the kernel of $\varrho^n$ by $F^{[n]}$ (as in [24]) we obtain for $m \geq m_0$ exact sequences

$$0 \longrightarrow F^{[n_m]} \xrightarrow{i} H_{n_m} \xrightarrow{\Psi} F_m \longrightarrow 0.$$

Let $0 \longrightarrow G \xrightarrow{i} \mathscr{D}' \xrightarrow{q} F \longrightarrow 0$ be an exact sequence as constructed in 2. Again lemma 5.3.4 gives $m_1 \geq m_0$ and a non-decreasing sequence $(p_m)_{m\geq m_1}$ such that every bounded set in $F^{[n_m]}$ is contained in the $q$-image of a bounded subset of

$$\prod_{l<p_m} \{0\} \times \prod_{l\geq p_m} s' =: B_{p_m}.$$

The continuity of $q$ implies that there are $m_2 \geq m_1$ and another non-decreasing sequence $(r_m)_{m\geq m_2}$ tending to infinity with $r_m \leq n_m$ and such that $q(B_{p_m}) \subseteq F^{[r_m]}$.

4. Now, we are going to construct an operator $R_m : F_m \longrightarrow A_{r_m} \subseteq \prod_{l\in\mathbb{N}} F_l$ satisfying $\Psi \circ R_m = j_m$ for $m \geq m_2$.

Since $F_m' \cong s$ is a reduced projective limit of spaces isomorphic to $c_0$ the dual of the canonical resolution for $F_m'$ is a topologically exact sequence

$$0 \longrightarrow \bigoplus_{l\in\mathbb{N}} X_l \xrightarrow{d} \bigoplus_{l\in\mathbb{N}} X_l \xrightarrow{\pi} F_m = \operatorname{ind}_l X_l \longrightarrow 0,$$

where $X_n \cong \ell_1$, $X_n \subseteq X_{n+1}$, $d((x_l)_{l\in\mathbb{N}}) = (x_l - x_{l-1})_{l\in\mathbb{N}}$, $x_0 = 0$, and $\pi((x_l)_{l\in\mathbb{N}}) = \sum_{l\in\mathbb{N}} x_l$ (more information about this representation of inductive limits in contained in chapter 6).

The unit vectors of $X_l \cong \ell_1$ are contained in the $\Psi$-image of a bounded subset of $H_{n_m}$, hence there are operators $r_l : X_l \longrightarrow H_{n_m}$ with $\Psi \circ r_l = id_{X_l}$

which implies $\Psi \circ (r_l - r_{l+1}|_{X_l}) = 0$. Therefore, there are $u_l : X_l \longrightarrow F^{[n_m]}$ with $i \circ u_l = r_l - r_{l+1}|_{X_l}$. The images under $u_l$ of the unit vectors in $X_l \cong \ell_1$ are again contained in the $q$-image of a bounded subset of $B_{p_m}$ which yields operators $\widetilde{u}_l : X_l \longrightarrow B_{p_m}$ with $q \circ \widetilde{u}_l = u_l$. Defining $\widetilde{u}((x_l)_{l\in\mathbb{N}}) := \sum_{l\in\mathbb{N}} \widetilde{u}_l(x_l)$ we obtain

$$\begin{array}{ccccccccc} & & B_{p_m} & & & & & & \\ & & \uparrow{\scriptstyle \widetilde{u}} & & & & & & \\ 0 & \longrightarrow & \bigoplus_{l\in\mathbb{N}} X_l & \longrightarrow & \bigoplus_{l\in\mathbb{N}} X_l & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

and because of $B_{p_m} \cong \mathscr{D}'$ and $\operatorname{Ext}^1_{PLS}(F, \mathscr{D}') = 0$ there is an extension $\widetilde{v} : \bigoplus_{l\in\mathbb{N}} X_l \longrightarrow B_{p_m}$, i.e. $\widetilde{v} \circ d = \widetilde{u}$.

For $x = (x_l)_{l\in\mathbb{N}} \in \bigoplus_{l\in\mathbb{N}} X_l$ we have $\widetilde{v}(x) = \sum_{l\in\mathbb{N}} \widetilde{v}_l(x_l)$ with $\widetilde{v}_l : X_l \longrightarrow B_{p_m}$, and from

$$\sum_{l\in\mathbb{N}} \widetilde{u}_l(x_l) = \widetilde{v} \circ d\left((x_l)_{l\in\mathbb{N}}\right) = \sum_{l\in\mathbb{N}} \left(\widetilde{v}_l - \widetilde{v}_{l+1}|_{X_l}\right)(x_l)$$

we deduce $\widetilde{u}_l = \widetilde{v}_l - \widetilde{v}_{l+1}|x_l$. This yields $q \circ \widetilde{v}_l - r_l = (q \circ \widetilde{v}_{l+1} - r_{l+1})|_{X_l}$, i.e.

$$R_m(x) = (r_l - i \circ q \circ \widetilde{v}_l)(x) \text{ for } x \in X_l \subseteq F_m$$

defines an operator $R_m : F_m \longrightarrow H_{r_m}$ with $\Psi \circ R_m = j_m$.

5. Finally, we choose $R_0 : \prod_{l<m_2} F_l \times \prod_{l\geq m_2} \{0\} \longrightarrow \prod_{l\in\mathbb{N}} F_l$ with $\Psi \circ R_0 = id$ and obtain a right inverse

$$R\left((y_m)_{m\in\mathbb{N}}\right) = R_0\left((y_1, \ldots, y_{m_2-1}, 0, \ldots)\right) + \sum_{m=m_2}^{\infty} R_m(y_m)$$

of $\Psi$, which converges since $R_m$ has values in $\prod_{l<r_m} \{0\} \times \prod_{l\geq r_m} F_l$ and $r_m$ tends to infinity.

We have shown that the canonical resolution

$$0 \longrightarrow F \longrightarrow \prod_{l\in\mathbb{N}} F_l \xrightarrow{\Psi} \prod_{l\in\mathbb{N}} F_l \longrightarrow 0$$

splits, hence $F$ is isomorphic to a complemented subspace of $\prod_{l\in\mathbb{N}} F_l \cong \mathscr{D}'$. □

**Corollary 5.3.6** *Let $E$ and $F$ be $(PLS)$-spaces.*

1. *If $E$ is isomorphic to a bornological subspace of $\mathscr{D}'$ then there exists an exact sequence*
$$0 \longrightarrow E \longrightarrow \mathscr{D}' \longrightarrow \mathscr{D}' \longrightarrow 0.$$

2. *If $F$ is isomorphic to a quotient of $\mathscr{D}'$ then there exists an exact sequence*

$$0 \longrightarrow \mathscr{D}' \longrightarrow \mathscr{D}' \longrightarrow F \longrightarrow 0.$$

**Proof.** Let $G$ be a (PLN)-space, i.e. a projective limit of $(LS)$-spaces $G_n$ which have even nuclear linking maps. Then there are a subspace $\widetilde{G}$ of $\mathscr{D}'$ and an exact sequence

$$0 \longrightarrow \mathscr{D}' \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 0.$$

Indeed, as we have seen in the proof of corollary 5.2.10, $G_n'$ is a quotient of a subspace of $s$, hence $G_n$ is a subspace of a quotient $H_n$ of $s'$. Dualizing the first part of 5.2.10 gives exact sequences

$$0 \longrightarrow s' \longrightarrow s' \longrightarrow H_n \longrightarrow 0$$

and taking products we obtain an exact sequence

$$0 \longrightarrow \mathscr{D}' \longrightarrow \mathscr{D}' \stackrel{q}{\longrightarrow} \prod_{n\in\mathbb{N}} H_n \longrightarrow 0.$$

$G$ is a closed subspace of $\prod_{n\in\mathbb{N}} G_n \subseteq \prod_{n\in\mathbb{N}} H_n$ and with $\widetilde{G} = q^{-1}(G)$ we obtain

$$0 \longrightarrow \mathscr{D}' \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 0.$$

Now the proof goes exactly as that of 5.2.10. If $E$ is a bornological subspace of $\mathscr{D}'$ and $\mathscr{E}$ a strongly reduced spectrum of (LN)-spaces with $\operatorname{Proj}\mathscr{E} \cong E$ we apply corollary 3.3.10 to get $\operatorname{Proj}^1\mathscr{E} = 0$ and this yields that $G := \mathscr{D}'/E$ is a projective limit of (LN)-spaces. The pull back construction gives a $(PLS)$-space $H$ and the following diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{D}' & \longrightarrow & \widetilde{G} & \longrightarrow & \mathscr{D}'/E & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{D}' & \longrightarrow & H & \longrightarrow & \mathscr{D}' & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & E & = & E & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The second row splits by 5.3.3, thus $H \cong \mathscr{D}' \times \mathscr{D}' \cong \mathscr{D}'$ and therefore, $\widetilde{G}$ is isomorphic to a quotient of $\mathscr{D}'$, hence to a complemented subspace of $\mathscr{D}'$. Taking the product of the first row (column) with

$$0 \longrightarrow \mathscr{D}' \stackrel{id}{\longrightarrow} \mathscr{D}' \longrightarrow 0 \longrightarrow 0 \quad (0 \longrightarrow 0 \longrightarrow \mathscr{D}' \stackrel{id}{\longrightarrow} \mathscr{D}' \longrightarrow 0)$$

and using the Pełczyński decomposition method as in 5.2.10 (with $(\mathscr{D}')^{\mathbb{N}} \cong \mathscr{D}'$ instead of $s(s)$) we find exact sequences

$$0 \longrightarrow \mathscr{D}' \longrightarrow \mathscr{D}' \longrightarrow \mathscr{D}'/E \longrightarrow 0 \text{ and } 0 \longrightarrow E \longrightarrow \mathscr{D}' \longrightarrow \mathscr{D}' \longrightarrow 0.$$

□

The proof of the analogon of corollary 5.2.9 in the category of $(PLS)$-spaces is based on the following construction of a "skew pull back" which can be carried out in any quasi-abelian category. As in the case of $(PLS)$-spaces, we write $\operatorname{Ext}^1(E,F) = 0$ if the equivalent conditions of proposition 5.1.3 hold.

**Proposition 5.3.7** *Let in a quasi-abelian category*

$$0 \longrightarrow F \xrightarrow{I} X \xrightarrow{Q} E \longrightarrow 0$$

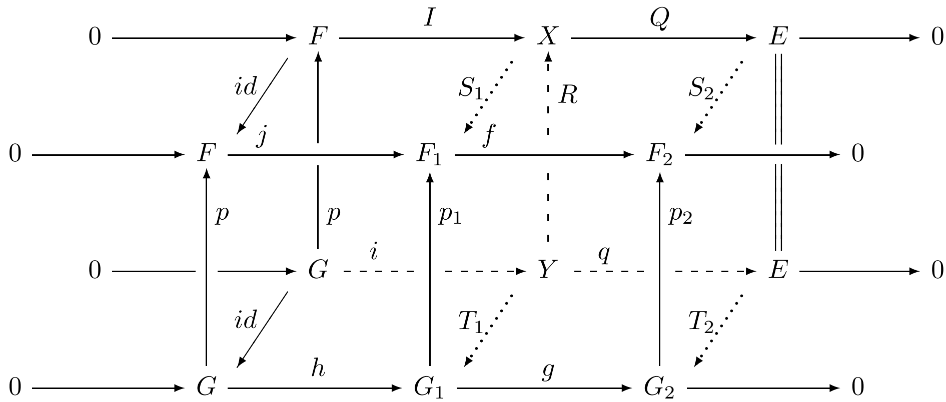
*be an exact sequence and* $p : G \longrightarrow F$ *such that there is a commutative diagram with exact rows and column*

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & F & \xrightarrow{j} & F_1 & \longrightarrow & F_2 & \longrightarrow & 0 \\
 & & \uparrow p & & \uparrow p_1 & & \uparrow p_2 & & \\
0 & \longrightarrow & G & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & H_2 & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

*If* $\operatorname{Ext}^1(E,F_1) = 0$ *and* $\operatorname{Ext}^1(E,H_2) = 0$, *then there exists a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{I} & X & \xrightarrow{Q} & E & \longrightarrow & 0 \\
 & & \uparrow p & & \uparrow R & & \| & & \\
0 & \longrightarrow & G & \xrightarrow{i} & Y & \xrightarrow{q} & E & \longrightarrow & 0
\end{array}$$

**Proof.** The proof can be visualized in the following diagram where we first ignore $Y$ and the dotted arrows. The remaining part is given by the assumptions, and we will construct $S_1, S_2, Y, T_1, T_2$, and finally $R$.




Because of $\operatorname{Ext}^1(E, F_1) = 0$ there is an extension $S_1 : X \longrightarrow F_1$ of $j : F \longrightarrow F_1$, and there is a unique $S_2 : E \longrightarrow F_2$ with $S_2 \circ Q = f \circ S_1$ since $f \circ S_1 \circ I = f \circ j = 0$. The exactness of

$$0 \longrightarrow H_2 \longrightarrow G_2 \longrightarrow F_2 \longrightarrow 0$$

together with $\operatorname{Ext}^1(E, H_2)$ imply that $S_2 : E \longrightarrow F_2$ has a lifting $T_2 : E \longrightarrow G_2$, i.e. $p_2 \circ T_2 = S_2$. Using the pull back construction we obtain an exact sequence

$$0 \longrightarrow G \xrightarrow{\;i\;} Y \xrightarrow{\;q\;} E \longrightarrow 0$$

and $T_1 : Y \longrightarrow G_1$ such that the lower plane of the diagram commutes. We will finally construct $R : Y \longrightarrow X$ such that the whole diagram is commutative.

Let the following diagram be the pull-back of $q : Y \longrightarrow E$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{\;I\;} & X & \xrightarrow{\;Q\;} & E & \longrightarrow & 0 \\
 & & \| & & \uparrow r & & \uparrow q & & \\
0 & \longrightarrow & F & \xrightarrow{\;\varphi\;} & Z & \xrightarrow{\;\pi\;} & Y & \longrightarrow & 0
\end{array}$$

Then $f \circ (S_1 \circ r - p_1 \circ T_1 \circ \pi) = S_2 \circ Q \circ r - p_2 \circ T_2 \circ q \circ \pi = S_2 \circ Q \circ r - S_2 \circ q \circ \pi = 0$, hence there is a unique $\psi : Z \longrightarrow F$ with $j \circ \psi = S_1 \circ r - p_1 \circ T_1 \circ \pi$. We define $\varrho = r - I \circ \psi : Z \longrightarrow X$ and obtain

$$(1) \quad Q \circ \varrho = Q \circ r = q \circ \pi \quad \text{and (2)} \quad S_1 \circ r - j \circ \psi = p_1 \circ T_1 \circ \pi.$$

We have $\psi \circ \varphi = id$ because $j$ is a monomorphism and

$$j \circ \psi \circ \varphi = S_1 \circ r \circ \varphi - p_1 \circ T_1 \circ \pi \circ \varphi = S_1 \circ I = j = j \circ id.$$

We obtain $\varrho \circ \varphi = r \circ \varphi - I = 0$ and thus, there is $R : Y \longrightarrow X$ with $R \circ \pi = \varrho$. Since $\pi$ is an epimorphism we obtain $Q \circ R = q$ and $S_1 \circ R = p_1 \circ T_1$ from (1) and (2).

It remains to show $I \circ p = R \circ i$. Since $Q \circ R \circ i = q \circ i = 0$ there is a unique $\widetilde{p} : G \longrightarrow F$ with $I \circ \widetilde{p} = R \circ i$. On the other hand we have $j \circ \widetilde{p} = S_1 \circ I \circ \widetilde{p} = S_1 \circ R \circ i = p_1 \circ T_1 \circ i = p_1 \circ h = j \circ p$ and this implies $p = \widetilde{p}$ as $j$ is a monomorphism. $\square$

The preceding result has a dual version about the existence of a "skew push out". One can either mimic the proof above or consider the opposite category $\mathscr{K}^{op}$ which has the same objects and reversed arrows. The pull back (push out) in $\mathscr{K}$ is then a push out (pull back) in $K^{op}$ and $\mathrm{Ext}^1_{\mathscr{K}}(E, F) = 0$ is equivalent to $\mathrm{Ext}^1_{\mathscr{K}^{op}}(F, E) = 0$. Since we do not need this skew push out we refrain from stating the result explicitely.

Combining 5.3.6 and 5.3.7 we obtain the main theorem of this section which appears in [71] and improves a result of Domański and Vogt [25] who showed $\mathrm{Ext}^1_{PLS}(E, \mathscr{D}') = 0$ and $\mathrm{Ext}^1_{PLS}(\mathscr{D}', F) = 0$ for a subspace $E$ of $\mathscr{D}'$ and a quotient $F$ of $\mathscr{D}'$.

**Theorem 5.3.8** *Let $E$ and $F$ be $(PLS)$-spaces such that $E$ is isomorphic to a subspace of $\mathscr{D}'$ and $F$ is isomorphic to a quotient of $\mathscr{D}'$. Then* $\mathrm{Ext}^1_{PLS}(E, F) = 0$.

**Proof.** Let $0 \longrightarrow F \longrightarrow X \longrightarrow E \longrightarrow 0$ be an exact sequence of $(PLS)$-spaces and

$$0 \longrightarrow \mathscr{D}' \longrightarrow \mathscr{D}' \overset{p}{\longrightarrow} F \longrightarrow 0$$

be an exact sequence obtained in corollary 5.3.6. Using lemma 5.3.2 we find strongly reduced spectra $\mathscr{A} = (A_n, \tau^n_m)$, $\mathscr{B} = (B_n, \sigma^n_m)$, and $\mathscr{F} = (F_n, \varrho^n_m)$ of $(LS)$-spaces such that $\mathrm{Proj}\, \mathscr{A} = \mathscr{D}'$, $A_n \cong s'$, $\mathrm{Proj}\, \mathscr{B} = \mathscr{D}'$, $\mathrm{Proj}\, \mathscr{F} = F$, each $F_n$ is a quotient of $s'$, and we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{D}' & \longrightarrow & \mathscr{D}' & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau^n} & & \downarrow{\scriptstyle \sigma^n} & & \downarrow{\scriptstyle \varrho^n} & & \\
0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & F_n & \longrightarrow & 0
\end{array}$$

If $f = \Psi_{\mathscr{F}} : \prod_n F_n \longrightarrow \prod_n F_n$ and $g = \Psi_{\mathscr{B}} : \prod_{n\in\mathbb{N}} B_n \longrightarrow \prod_{n\in\mathbb{N}} B_n$ are the canonical maps (which are surjective since $\mathrm{Proj}^1 \mathscr{F} = 0$ and $\mathrm{Proj}^1 \mathscr{B} = 0$) we obtain the following diagram with exact rows and column

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & F & \longrightarrow & \prod\limits_{n\in\mathbb{N}} F_n & \xrightarrow{f} & \prod\limits_{n\in\mathbb{N}} F_n & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{D}' & \longrightarrow & \prod\limits_{n\in\mathbb{N}} B_n & \xrightarrow{g} & \prod\limits_{n\in\mathbb{N}} B_n & \longrightarrow & 0\\
 & & & & & & \uparrow & & \\
 & & & & & & \prod\limits_{n\in\mathbb{N}} A_n & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

From 5.3.3 we deduce $\mathrm{Ext}^1_{PLS}(E, \prod\limits_{n\in\mathbb{N}} F_n) = 0$ and $\mathrm{Ext}^1_{PLS}(E, \prod\limits_{n\in\mathbb{N}} A_n) = 0$, hence we can apply proposition 5.3.7 to find a skew pull back diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & X & \longrightarrow & E & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle p} & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathscr{D}' & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

The second row spits again by 5.3.3 hence also the first row splits. □

The theorem above is rather abstract, intrinsic characterizations of subspaces and quotients of $\mathscr{D}'$ can be found in [25, 24]. However, from 5.3.8 one can easily deduce the splitting result [24, theorem 1] for distributional complexes: if

$$0 \longrightarrow F \longrightarrow F_0 \xrightarrow{T_0} F_1 \xrightarrow{T_1} F_2 \xrightarrow{T_2} F_3 \longrightarrow \cdots$$

is an exact complex of $(PLS)$-spaces with $F_n \cong \mathscr{D}'$ for all $n \in \mathbb{N}_0$, then for every $n \geq 1$ the operator $T_n : F_n \longrightarrow \mathrm{im}(T_n)$ has a right inverse. Indeed, $\mathrm{im}(T_n) = \ker(T_{n+1})$ is isomorphic to a subspace of $\mathscr{D}'$ and $\ker(T_n) = \mathrm{im}(T_{n-1})$ is isomorphic to a quotient of $\mathscr{D}'$, hence theorem 5.3.8 implies that

$$0 \longrightarrow \ker(T_n) \longrightarrow F_n \longrightarrow \mathrm{im}(T_n) \longrightarrow 0$$

splits. The same reasoning gives a right inverse of $T_0 : F_0 \longrightarrow \mathrm{im}(T_0)$ if $F$ is isomorphic to a quotient of $\mathscr{D}'$. Here is would be enough to require that $F_1$ is isomorphic to a subspace of $\mathscr{D}'$. For more information and applications we refer to [24].

# 6

# Inductive spectra of locally convex spaces

Inductive limits have been treated extensively in the literature, and the aim of this chapter is not to consider all the different aspects of this theory, the interested reader is referred e.g. to the introductory text [5] of Bierstedt, to the article [27] of Floret, and the book of Bonet and Pérez-Carreras [51], in particular to chapter 8. We will prove some results about (LF)-spaces relating acyclicity – which reflects properties of the dual projective spectrum – to regularity conditions and completeness and will then concentrate on two questions of Palamodov about inductive limits of general locally convex spaces.

The homological character of the inductive limit functor is very different from that of the projective limit functor. For instance, it is exact as a functor acting on inductive spectra of vector spaces. As a functor on inductive spectra of locally convex spaces it transforms short exact sequences into acyclic sequences but monohomomorphisms are in general not transformed into homomorphisms. To measure this lack of exactness Palamodov constructed the first satellite of a contravariant functor associated with ind which plays a similar role as derived functors do. We will not repeat this construction here because our contribution to the unsolved problems [50, §12.3 and 4] can be understood easily without it.

We confine ourselves to countable and injective inductive spectra. Thus, by an inductive spectrum $(X_n)_{n\in\mathbb{N}}$ we always mean a sequence $X_n$ of locally convex spaces with $X_n \subset X_{n+1}$ and continuous inclusion maps. The inductive limit is then $X = \operatorname{ind} X_n = \bigcup_{n\in\mathbb{N}} X_n$ endowed with the finest locally convex topology such that all inclusions $X_n \hookrightarrow X$ are continuous. An absolutely convex set is a neighbourhood of 0 in the inductive limit if and only if its intersection with each "step" $X_n$ belongs to $\mathscr{U}_0(X_n)$. A basis of $\mathscr{U}_0(X)$ is thus given by

$$\left\{ \bigcup_{m\in\mathbb{N}} \sum_{n\leq m} U_n : U_n \in \mathscr{U}_0(X_n) \right\}.$$

J. Wengenroth: LNM 1810, pp. 109–118, 2003.

Given an inductive spectrum $(X_n)_n$ there is a canonical algebraically exact sequence

$$0 \longrightarrow \bigoplus_{n\in\mathbb{N}} X_n \xrightarrow{d} \bigoplus_{n\in\mathbb{N}} X_n \xrightarrow{\sigma} X \longrightarrow 0$$

where $d((x_n)_{n\in\mathbb{N}}) = (x_n - x_{n-1})_{n\in\mathbb{N}}$, $x_0 = 0$, and $\sigma((x_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} x_n$.

$d$ and $\sigma$ are continuous, and $\sigma$ is also open. The spectrum is called (weakly) acyclic if $d$ is (weakly) open onto its range.

This definition is related to the question whether a subspace $L$ of an inductive limit $\operatorname{ind} X_n$ is topologically the inductive limit of the spaces $L_n = X \cap L_n$ (in this case $L$ is called a limit subspace) or if it has at least the same topological dual as the inductive limit (then $L$ is called well-located). Using either the duality explained below or a diagram chase on gets:

> A subspace $L$ of the limit $X = \operatorname{ind} X_n$ of a (weakly) acyclic inductive spectrum is (well-located) a limit subspace if and only if the spectrum $(X_n/L_n)_{n\in\mathbb{N}}$ is (weakly) acyclic.

An easy calculation shows that the transposed of $d$ is

$$\Psi : \prod_{n\in\mathbb{N}} X_n' \longrightarrow \prod_{n\in\mathbb{N}} X_n',\ (f_n)_{n\in\mathbb{N}} \longmapsto (f_n - \varrho_{n+1}^n(f_{n+1}))_{n\in\mathbb{N}}$$

where $\varrho_{n+1}^n : X_{n+1}' \longrightarrow X_n'$ is the restriction (i.e. the transposed of the inclusion). Thus, there is a close connection between (weakly) acyclic inductive spectra and the derived projective limit functor. In particular, an inductive spectrum $(X_n)_{n\in\mathbb{N}}$ is weakly acyclic if and only if $\operatorname{Proj}^1 \mathscr{Y} = 0$ where $\mathscr{Y} = (X_n', \varrho_m^n)$ with the restrictions as spectral maps. Thus, the results of chapter 3 have immediate counterparts for weakly acyclic spectra which we do not state explicitely.

There is also a direct relation between acyclic inductive spectra and properties of projective limits. Using theorem 2.2.2 we obtain that $(X_n)_{n\in\mathbb{N}}$ is acyclic if and only if for every set $I$ we have $\operatorname{Proj}^1(L(X_n, \ell_I^\infty), R_m^n) = 0$ where $R_{n+1}^n(T) = T \circ \varrho_{n+1}^n$. The spaces $L(X_n, \ell_I^\infty)$ are covered by the system $\{(U^\circ)^I : U \in \mathscr{U}_0(X_n)\}$ (here, we identify $T \in L(X_n, \ell_I^\infty)$ with a family $(T_i)_{i\in I}$) and using theorem 3.2.14 and the remark following its proof we obtain a general sufficient condition for acyclicity (a direct proof of this can be obtained as in [69]).

**Theorem 6.1** *An inductive spectrum $(X_n)_{n\in\mathbb{N}}$ is acyclic if it satisfies*

$$\forall\ n \in \mathbb{N}\ \exists\, m \geq n\ \forall\ k \geq m\ \exists\, U \in \mathscr{U}_0(X_n)\ \forall\ V \in \mathscr{U}_0(X_m)\ \exists\, W \in \mathscr{U}_0(X_k)$$

$$W \cap U \subseteq V$$

The condition of this theorem means that the topologies of $X_m$ and $X_k$ coincide on some 0-neighbourhood of $X_n$. To see this one only needs that two locally convex topologies coincide on an absolutely convex set if they induce the same 0-neighbourhood filter. Note that the topologies then also induce the same uniformity.

The next proposition is obtained from the definition just by rephrasing the openness in terms of 0-neighbourhoods. We use expressions like $\sum\limits_{k\in\mathbb{N}} V_k$ as an abbreviation for $\bigcup\limits_{m\in\mathbb{N}} \sum\limits_{k=1}^{m} V_k$.

**Proposition 6.2** *An inductive spectrum $(X_n)_{n\in\mathbb{N}}$ is acyclic if and only if*

$$\forall\, (U_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{U}_0(X_n) \ \exists\, (V_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{U}_0(X_n) \ \forall\, n\in\mathbb{N}$$

$$\sum_{k\le n} V_k \cap \sum_{k>n} V_k \subseteq U_n.$$

A first consequence of this result is that the limit of an acyclic spectrum is separated [50, proposition 7.1]. If the spaces $X_n$ are complete, this can be deduced directly from the definition since then $X$ is a quotient of of the direct sum by a complete hence closed subspace.

**Proposition 6.3** *The limit of an acyclic inductive spectrum of separated locally convex spaces is again separated.*

**Proof.** For $0\ne x\in \operatorname{ind} X_n$ there are $U_n\in\mathscr{U}_0(X_n)$ with $x\notin U_n$. We choose $V_n\in\mathscr{U}_0(X_n)$ with $\sum\limits_{k\le n} V_k \cap \sum\limits_{k>n} V_k \subseteq \frac{1}{n}U_n$ for all $n\in\mathbb{N}$. The sequence $n\sum\limits_{k\le n} V_k$ covers $\operatorname{ind} X_n$, hence, $x\in n\sum\limits_{k\le n} V_k$ for some $n\in\mathbb{N}$ and thus $x$ does not belong to $\sum\limits_{k>n} V_k$ which is a 0-neighbourhood in $\operatorname{ind} X_n$. □

We will now concentrate on (LF)-spaces, i.e. inductive limits of Fréchet spaces. Let us first note that two inductive spectra of Fréchet spaces $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ with the same inductive limit are equivalent, i.e. there are sequences $m(n)$ and $k(n)$ such that $X_n\subseteq Y_{m(n)}\subseteq X_{k(n)}$ with continuous inclusions. This is a special case of Grothendieck's factorization theorem. The spectra formed by the spaces $L(X_n,\ell_I^\infty)$ and $L(Y_n,\ell_I^\infty)$ are then equivalent projective spectra and we obtain that $(X_n)_{n\in\mathbb{N}}$ is acyclic if and only if $(Y_n)_{n\in\mathbb{N}}$ is acyclic. This justifies to call the inductive limit acyclic if some defining spectrum is acyclic.

A regularity condition of an inductive limit $X=\operatorname{ind} X_n$ is a requirement that each subset of the inductive limit of a certain class $\mathscr{C}_1(X)$ should be contained in some step $X_n$ and belong to another class $\mathscr{C}_2(X_n)$ (one might speak of $(\mathscr{C}_1,\mathscr{C}_2)$-regularity). $(\mathscr{B},\mathscr{B})$-regularity is then the classical regularity defined by Makarov [43].

An inductive spectrum is called sequentially retractive [26] if it is $(c_0,c_0)$-regular and compactly reactive if it is $(\mathscr{K},\mathscr{K})$-regular. It is called boundedly stable if on each set which is bounded in some $X_n$ all but finitely many of the step topologies coincide. Finally, it is called boundedly retractive if it is regular and on each bounded set the inductive limit topology coincides with some step topology. We have the following theorem from [69]. The equivalence of the first two conditions is due to Palamodov [50, theorems 6.1 and 6.2] and Retakh [54] who called the second condition $(M)$. The equivalence of 5. and 6. was proved with completely different methods by B. Cascales and J. Orihuela [20]. The inequalities in condition 7. are dual to condition $(P_3)$ defined in 3.2.17 and appeared already in 3.3.7.

**Theorem 6.4** *For an (LF)-space* $(X, \mathscr{T}) = \operatorname{ind}(X_n, \mathscr{T}_n)$ *the following conditions are equivalent.*

1. *$X$ is acyclic.*
2. *There are absolutely convex $U_n \in \mathscr{U}_0(X_n)$ with $U_n \subseteq U_{n+1}$ and for each $n \in \mathbb{N}$ exists $m \geq n$ such that $\mathscr{T}$ and $\mathscr{T}_m$ coincide on $U_n$.*
3. $\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\, U \in \mathscr{U}_0(X_n)\quad \mathscr{T}_k$ *and* $\mathscr{T}_m$ *coincide on* $U$.
4. *$X$ is boundedly retractive.*
5. *$X$ is compactly retractive.*
6. *$X$ is sequentially retractive.*
7. *$X$ is boundedly stable and satisfies the following condition $(P_3^*)$:*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\, N \in \mathbb{N}\ \forall\, M \in \mathbb{N}\ \exists\, K \in \mathbb{N},\, S > 0$$

$$\forall x \in X_n \quad \|x\|_{m,M} \leq S\,(\|x\|_{k,K} + \|x\|_{n,N})$$

*where $(\|\cdot\|_{n,N})_{N\in\mathbb{N}}$ is a fundamental sequence of seminorms for $X_n$.*

**Proof.** The equivalence of the first three conditions follows from the relation to the Proj-functor explained above, theorem 3.2.14 and corollary 3.3.15. 2. implies 4. since $X$ is regular which is proved in 6.5 below. It is simple to deduce 5. from 4. and 6. from 5. Let us assume that $X$ is sequentially retractive and that 3. fails for some $n \in \mathbb{N}$. Then there are $k(m) \geq m$ such that $\mathscr{T}_m$ and $\mathscr{T}_{k(m)}$ do not coincide on any 0-neighbourhood $U$ of $X_n$. Choosing a decreasing basis $(U_m)_{m\in\mathbb{N}}$ of $\mathscr{U}_0(X_n)$ and using the metrizability of $X_m$ and $X_{k(m)}$ we find sequences $(x_j^m)_{j\in\mathbb{N}}$ in $U_m$ converging to 0 in $X_{k(m)}$ (hence also in $X$) which do not converge to 0 in $X_m$. We order $(x_j^m)_{m,j\in\mathbb{N}}$ in any way to a single-indexed sequence $(y_i)_{i\in\mathbb{N}}$ and obtain $(y_i)_{i\in\mathbb{N}} \in c_0(X)$. Indeed, given $U \in \mathscr{U}_0(X)$ there is $m_0 \in \mathbb{N}$ with $U_m \subseteq U$ for all $m \geq m_0$ (since $(U_m)_{m\in\mathbb{N}}$ is a decreasing basis) and we find $j_0 \in \mathbb{N}$ such that $x_j^m \in U$ for $1 \leq m \leq m_0$ and $j \geq j_0$. Hence all but finitely many $y_i$ belong to $U$. Since $X$ is sequentially retractive $(y_i)_{i\in\mathbb{N}}$ converges to 0 in some $X_m$ and then also the subsequence $(x_j^m)_{j\in\mathbb{N}}$ tends to 0 in $X_m$ contradicting the choice of $x_j^m$.

3. easily implies 7. and it remains to prove the converse implication. Let us first show that bounded stability yields that $\forall\, m \in \mathbb{N}\ \exists\, p \geq m\ \ \forall\, k \geq p$ the topologies $\mathscr{T}_p$ and $\mathscr{T}_k$ coincide on every bounded subset of $X_m$.

Otherwise there would be $B_p \in \mathscr{B}(X_m)$ and $k(p) \geq p$ such that $\mathscr{T}_p$ and $\mathscr{T}_{k(p)}$ do not coincide on $B_p$ hence neither on $\varepsilon_p B_p$ for all $\varepsilon_p > 0$ contradicting the bounded stability if $\varepsilon_p$ are chosen such that $B = \bigcup_{p\geq m} \varepsilon_p B_p$ is bounded in $X_m$ (which is easily done using the metrizability).

Let now $n \leq m \leq k$ as in 7. The inequalities for the seminorms imply that every subset of $U = \{x \in X_n : \|x\|_{n,N} \leq 1\}$ which is bounded in $X_k$ is also bounded in $X_m$. If $p \geq m$ is chosen as above and $k \geq p$ then $\mathscr{T}_p$ and $\mathscr{T}_k$ coincide on $U$ since every sequence in $U$ which converges in $X_k$ is bounded in $X_m$ hence it converges to the same limit in $X_p$ as $\mathscr{T}_p$ and $\mathscr{T}_k$ coincide on bounded subsets of $X_m$. □

The next corollary is again due to Palamodov [50, corollaries 7.1 and 7.2].

**Corollary 6.5** *Acyclic (LF)-spaces are complete and regular.*

**Proof.** We will use a sequence $U_n$ of 0-neighbourhoods as in 2. of the preceding theorem. Since $\mathscr{T}$ and $\mathscr{T}_m$ induce the same uniformity on $U_n$ and $(X_m, \mathscr{T}_m)$ is complete we get that the closure of $U_n$ in the completion $\widetilde{X}$ is contained in $X_m$. It remains to show that every $x \in \widetilde{X}$ is contained in $\overline{nU_n}^{\widetilde{X}}$ for some $n \in \mathbb{N}$. Assuming the contrary we find $V_n \in \mathscr{U}_0(X)$ with $x \notin nU_n + 2\overline{V_n}^{\widetilde{X}}$. The set $W = \bigcap_{n\in\mathbb{N}}(U_n + V_n)$ is a 0-neighbourhood in $X$ since $W \cap X_n$ contains $V_1 \cap \cdots \cap V_{n-1} \cap U_n \in \mathscr{U}_0(X_n)$ (here we used $U_n \subseteq U_{n+1}$). Hence there is $y \in X$ with $x - y \in \overline{W}^{\widetilde{X}}$. Since $X = \bigcup_{n\in\mathbb{N}} nU_n$ we find $n \in \mathbb{N}$ with $y \in (n-1)U_n$ and obtain the contradiction

$$x \in (n-1)U_n + \overline{W}^{\widetilde{X}} \subseteq (n-1)U_n + \overline{U_n + V_n}^{\widetilde{X}} \subseteq nU_n + 2\overline{V_n}^{\widetilde{X}}.$$

We have shown that $X$ is complete and since $X$ is separated we get the regularity from Grothendieck's factorization theorem. □

We can prove regularity also without Grothendieck's factorization theorem. Indeed, with the same notation as above we have $\overline{U_n}^{X} = \overline{U_n}^{X_m} \subseteq U_m$. Since $X$ is complete each bounded set is contained in a closed Banach disc $B$ which is covered by the closed sets $n\overline{U_n}^{X}$ and Baire's theorem implies that some of these sets contains interior points (with respect to the Banach space topology on $[B]$). This implies $B \subset S\overline{U_n}^{X} \subset SU_m$ for some $S > 0$ and hence the boundedness of $B$ in some $X_k$.

This shows that corollary 6.5 holds for inductive spectra of complete spaces (without any metrizability assumption) whenever the equivalent conditions 2. and 3. of 6.4 are satisfied.

Palamodov asked in [50, §12.3 and 4] whether corollary 6.5 is true for all acyclic spectra of complete locally convex spaces. We will provide a positive result below under a rather weak additional assumption. But let us first make some more remarks about (LF)-spaces and condition $(P_3^*)$. The most obvious case of boundedly stable (LF)-spaces is that of limits of Fréchet-Montel spaces (since on a compact set there is no coarser separated topology). In this case $(P_3^*)$ characterizes acyclicity. We have already shown in 3.3.7 that regular

(LF)-spaces satisfy $(P_3^*)$ (even in a slightly stronger form where $N$ is independent of $k$), and the proof used only $\beta$-regularity, i.e. every bounded set in the inductive limit which is contained in some step is also bounded in some step. An inductive limit is $\alpha$-regular if every bounded set in the inductive limit is contained in some step. The next result is again taken from [69].

**Proposition 6.6** *Every $\alpha$-regular inductive limit satisfies $(P_3^*)$.*

**Proof.** Let us first show that for each $n \in \mathbb{N}$ there are $U \in \mathscr{U}_0(X_n)$ and $m \geq n$ with $\overline{U}^X \subseteq X_m$. We assume the contrary and take a decreasing basis $(U_m)_{m\in\mathbb{N}}$ of $\mathscr{U}_0(X_n)$. Then there are $x_m \in \overline{U_m}^X \setminus X_m$. But $(x_m)_{m\in\mathbb{N}}$ converges to 0 in $X$ (since for each $U \in \mathscr{U}_0(X)$ all but finitely many $U_m$ are contained in $U$) which contradicts the $\alpha$-regularity.

We now prove that every subset $B$ of $U$ which is bounded in some $X_k$ is also bounded in $X_m$. Indeed, $D = \overline{\Gamma(B)}^{X_k}$ is a Banach disc which is contained in $X_m$ such that the inclusion $[D] \hookrightarrow X_m$ has closed graph, hence it is continuous which implies that $D$ is bounded in $X_m$. Finally, this implies $(P_3^*)$ as the proof of 3.3.7 shows. □

We obtain that for spectra of Fréchet-Montel spaces every reasonable regularity condition is already equivalent to acyclicity. We state only few of the numerous characterizations.

**Corollary 6.7** *Let $X = \operatorname{ind} X_n$ be an inductive limit of Fréchet-Montel spaces. Then the conditions 1.-7. of theorem 6.4 are equivalent to each of the following:*

8. *$X$ is $\alpha$-regular.*
9. *$X$ is $\beta$-regular.*
10. *$X$ is regular.*
11. *$X$ is complete and Hausdorff.*

The rest of this chapter is devoted to Palamodov's question whether corollary 6.5 is true for acyclic inductive spectra of complete locally convex spaces. We first provide a description of the completion of an acyclic inductive limit which enables us to derive positive solutions under a very weak additional assumption.

In the proofs below we will freely use the fact that for a separated locally convex space $X$, $U \in \mathscr{U}_0(X)$, $A \subseteq X$ absolutely convex, and $\varepsilon > 0$ we have

$$\overline{A}^{\widetilde{X}} \cap \overline{U}^{\widetilde{X}} \subseteq (1+\varepsilon)\overline{A \cap U}^{\widetilde{X}}.$$

Indeed, since $\overline{U}^{\widetilde{X}}$ is a 0-neighbourhood in $\widetilde{X}$ we have for each $V \in \mathscr{U}_0(\widetilde{X})$

$$\overline{A}^{\widetilde{X}} \cap \overline{U}^{\widetilde{X}} \subseteq \left(A + \frac{\varepsilon}{2}\left(\overline{U}^{\widetilde{X}} \cap V\right)\right) \cap \overline{U}^{\widetilde{X}}$$
$$\subseteq \left(A \cap \left(1 + \frac{\varepsilon}{2}\right)\overline{U}^{\widetilde{X}}\right) + V$$
$$\subseteq (1+\varepsilon)(A \cap U) + V.$$

**Proposition 6.8** *Let $(X_n)_{n\in\mathbb{N}}$ be an acyclic inductive spectrum of separated spaces and $X = \operatorname{ind} X_n$. We endow*

$$Z_n := \bigcap \left\{ [\overline{U}^{\widetilde{X}}] : U \in \mathscr{U}_0(X_n) \right\}$$

*with the locally convex topology having $\{Z_n \cap \overline{U}^{\widetilde{X}} : U \in \mathscr{U}_0(X_n)\}$ as a basis of $\mathscr{U}_0(Z_n)$. Then $(Z_n)_{n\in\mathbb{N}}$ is an acyclic and regular inductive spectrum of complete spaces such that $\widetilde{X} = \operatorname{ind} Z_n$ holds topologically.*

**Proof.** It is easily seen that the limit in $\widetilde{X}$ of a Cauchy net in $Z_n$ belongs to $Z_n$ and is the limit in $Z_n$ of the net, hence the spaces $Z_n$ are complete, and we obviously have $Z_n \subseteq Z_{n+1} \subseteq \widetilde{X}$ with continuous inclusions.

We first show that every bounded set $B \subset \widetilde{X}$ is contained and bounded in some $Z_n$. Assuming the contrary we find $(U_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{U}_0(X_n)$ such that $B$ is not absorbed by any $\overline{U_n}^{\widetilde{X}}$. We choose $(V_n)_n \in \prod_{n\in\mathbb{N}} \mathscr{U}_0(X_n)$ according to proposition 6.2. Then there is $n \in \mathbb{N}$ with $B \subseteq n\overline{\sum_{k\leq n} V_k}^{\widetilde{X}}$. Indeed, otherwise there would be $x_n \in \frac{1}{n}B$ and $W_n \in \mathscr{U}_0(\widetilde{X})$ with $x_n \notin \sum_{k\leq n} V_k + W_n$.

$$W = \bigcap_{n\in\mathbb{N}} \sum_{k\leq n} V_k + W_n$$

is a 0-neighbourhood in $\widetilde{X}$ since $\overline{W}^{\widetilde{X}} \subseteq 2W$ and $W \cap X_n$ contains the 0-neighbourhood $W_1 \cap \ldots \cap W_n \cap V_n$, and the boundedness of $B$ implies that there is $n \in \mathbb{N}$ such that $x_n \in \frac{1}{n}B \subseteq W \subseteq \sum_{k\leq n} V_k + W_n$.

If $B$ be contained in $n\overline{\sum_{k\leq n} V_k}^{\widetilde{X}}$ we choose $S > n$ with $B \subseteq S\overline{\sum_{k>n} V_k}^{\widetilde{X}}$ (which is possible since the latter set is a 0-neighbourhood in $\widetilde{X}$). Then

$$B \subseteq S\left(\overline{\sum_{k\leq n} V_k}^{\widetilde{X}} \cap \overline{\sum_{k>n} V_k}^{\widetilde{X}}\right) \subseteq (S+1)\overline{\left(\sum_{k\leq n} V_k \cap \sum_{k>n} V_k\right)}^{\widetilde{X}} \subseteq (S+1)\overline{U_n}^{\widetilde{X}}$$

which contradicts the choice of $U_n$.

What we have shown implies that $\widetilde{X} = \operatorname{ind} Z_n$ holds algebraically and that $(Z_n)_{n\in\mathbb{N}}$ is regular. The acyclicity of that spectrum follows from

$$\sum_{k\le n}\left(\overline{V_k}^{\widetilde{X}}\cap Z_k\right)\cap\sum_{k>n}\left(\overline{V_k}^{\widetilde{X}}\cap Z_k\right)\subseteq Z_n\cap\left(\overline{\sum_{k\le n}V_k}^{\widetilde{X}}\cap\overline{\sum_{k>n}V_k}^{\widetilde{X}}\right)$$

$$\subseteq Z_n\cap 2\overline{\sum_{k\le n}V_k\cap\sum_{k>n}V_k}^{\widetilde{X}}.$$

We finally have to show that the identity map $\operatorname{ind} Z_n \longrightarrow \widetilde{X}$ (which is clearly continuous) is open. Given $U\in\mathscr{U}_0(\operatorname{ind} Z_n)$ there are $U_n\in\mathscr{U}_0(X_n)$ with $\sum_{n\in\mathbb{N}}\overline{U_n}^{\widetilde{X}}\cap Z_n\subseteq U$. We choose $(V_n)_{n\in\mathbb{N}}\in\prod_{n\in\mathbb{N}}\mathscr{U}_0(X_n)$ with $V_n\subseteq\frac{1}{4}U_n$ and $\sum_{k\le n}V_k\cap\sum_{k>n}V_k\subseteq\frac{1}{2(n+3)}U_n$ and claim that $V:=\overline{\sum_{k\in\mathbb{N}}V_k}^{\widetilde{X}}$ is contained in $U$. Given $x\in V$ there is $n\in\mathbb{N}$ with $x\in 2\overline{\sum_{k\le n}V_k}^{\widetilde{X}}$ because otherwise there would be $W_n\in\mathscr{U}_0(\widetilde{X})$ with $x\notin 2\sum_{k\le n}V_k+W_n$ and since $W=\bigcap_{n\in\mathbb{N}}\sum_{k\le n}V_k+W_n$ is a 0-neighbourhood in $\widetilde{X}$ this contradicts

$$x\in\bigcup_{n\in\mathbb{N}}\sum_{k\le n}V_k+W\subseteq\bigcup_{n\in\mathbb{N}}2\sum_{k\le n}V_k+W_n.$$

We know already $\widetilde{X}=\bigcup_{n\in\mathbb{N}}Z_n$, hence we may enlarge $n$ such that $x\in Z_n$ and $x\in n\sum_{k\le n}\overline{V_k}^{\widetilde{X}}$. With $W=\overline{\sum_{k>n}V_k}^{\widetilde{X}}\in\mathscr{U}_0(\widetilde{X})$ we obtain

$$x\in 2\overline{\sum_{k\le n}V_k}^{\widetilde{X}}\cap n\sum_{k\le n}\overline{V_k}^{\widetilde{X}}\subseteq\left(2\sum_{k\le n}V_k+W\right)\cap n\sum_{k\le n}\overline{V_k}^{\widetilde{X}}$$

$$\subseteq 2\sum_{k\le n}V_k+\left(W\cap(n+2)\sum_{k\le n}\overline{V_k}^{\widetilde{X}}\right)$$

$$\subseteq 2\sum_{k\le n}V_k+(n+3)\overline{\left(\sum_{k\le n}V_k\cap\sum_{k>n}V_k\right)}^{\widetilde{X}}$$

$$\subseteq\frac{1}{2}\sum_{k\le n}U_k+\frac{1}{2}\overline{U_n}^{\widetilde{X}}\subseteq\sum_{k<n}U_k+\overline{U_n}^{\widetilde{X}},$$

and because of $x\in Z_n$ this yields $x\in\sum_{k<n}U_k+(\overline{U_n}^{\widetilde{X}}\cap Z_n)\subseteq U$. □

**Definition 6.9** *An inductive spectrum $(X_n)_{n\in\mathbb{N}}$ has local closed neighbourhoods if* $\forall\, n\in\mathbb{N}\ \exists\, m\ge n\ \forall\, U\in\mathscr{U}_0(X_m)\ \exists\, V\in\mathscr{U}_0(X_n)$

$\overline{V} \cap X_n \subseteq U$ *where the closure is taken in the limit* $\operatorname{ind} X_n$.

This notion has been introduced by Vogt [65] who showed that weakly acyclic (LF)-spaces have local closed neighbourhoods. The condition is always satisfied if there is a separated topological vector space $Y$ such that each $X_n$ is continuously embedded into $Y$ and has a basis of $\mathscr{U}_0(X_n)$ consisting of sets which are closed in the relative topology induced by $Y$. In this case, one can take $m = n$ in definition 6.9. In particular, this remark applies if the steps $X_n$ are projective limits of weighted function or sequence spaces.

**Theorem 6.10** *Let* $(X_n)_{n\in\mathbb{N}}$ *be an acyclic inductive spectrum with local closed neighbourhoods. If all steps are complete and separated then* $(X_n)_{n\in\mathbb{N}}$ *is regular and* $\operatorname{ind} X_n$ *is complete.*

**Proof.** Let $B$ be a bounded set in $\widetilde{X}$. We show that $B$ is contained and bounded in some $X_n$. Because of proposition 6.8 there is $n_0 \in \mathbb{N}$ such that $B$ is a bounded subset of $Z_{n_0}$. We claim that there is $n \geq n_0$ such that $(x + \overline{U}^{\widetilde{X}}) \cap X_n \neq \emptyset$ for all $x \in B$ and $U \in \mathscr{U}_0(X_n)$. Assuming the contrary we find $(U_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{U}_0(X_n)$ and $(x_n)_{n\in\mathbb{N}} \in B^{\mathbb{N}}$ with $\left(x_n + \overline{U}_n^{\widetilde{X}}\right) \cap X_n = \emptyset$. 6.2 gives $(V_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathscr{U}_0(X_n)$ with $\sum\limits_{k\leq n} V_k \cap \sum\limits_{k>n} V_k \subseteq \frac{1}{2n+1} U_n$. As we have seen in the proof of 6.8 there is $n \in \mathbb{N}$ with $B \subseteq n\overline{\sum\limits_{k\leq n} V_k}^{\widetilde{X}}$. Hence we can find $y_n \in n \sum\limits_{k\leq n} V_k$ with $x_n - y_n \in W = \overline{\sum\limits_{k>n} V_k}^{\widetilde{X}}$. This implies

$$x_n - y_n \in 2n\overline{\sum_{k\leq n} V_k}^{\widetilde{X}} \cap W \subseteq (2n+1)\overline{\sum_{k\leq n} V_k \cap \sum_{k>n} V_k}^{\widetilde{X}} \subseteq \overline{U}_n^{\widetilde{X}}$$

and thus $y_n = x_n - (x_n - y_n) \in (x_n + \overline{U}_n^{\widetilde{X}}) \cap X_n$, a contradiction.

For each $x \in B$ we have a filter basis

$$\mathscr{F}_x = \left\{ \left(x + \overline{U}^{\widetilde{X}}\right) \cap X_n : U \in \mathscr{U}_0(X_n) \right\}$$

in $X_n$ and the filter generated by $\mathscr{F}_x$ in $\widetilde{X}$ clearly converges to $x$. We choose $m \geq n$ according to the definition of local closed neighbourhoods. Then $\mathscr{F}_x$ is the basis of a Cauchy filter in $X_m$. Indeed, given $V \in \mathscr{U}_0(X_m)$ we choose $U \in \mathscr{U}_0(X_n)$ with $\overline{U}^X \cap X_n \subseteq \frac{1}{2}V$ and obtain

$$\left(x + \overline{U}^{\widetilde{X}}\right) \cap X_n - \left(x + \overline{U}^{\widetilde{X}}\right) \cap X_n \subseteq 2\overline{U}^{\widetilde{X}} \cap X_n = 2\overline{U}^X \cap X_n \subseteq V.$$

Now the completeness of $X_m$ implies that $\mathscr{F}_x$ converges to some $y(x)$ in $X_m$. Because of proposition 6.3, $\widetilde{X}$ is separated which implies $x = y(x) \in X_m$. Let us finally show that $B$ is bounded in $X_m$. Given $V \in \mathscr{U}_0(X_m)$ we choose again $U \in \mathscr{U}_0(X_n)$ with $\overline{U}^X \cap X_n \subseteq V$. Since $B$ is bounded in $Z_{n_0} \subseteq Z_n$ there is $S > 0$ with $B \subseteq S\overline{U}^{\widetilde{X}}$. For every $x \in B$ this implies

$$\left(x + \overline{U}^{\widetilde{X}}\right) \cap X_n \subseteq (S+1)\overline{U}^{\widetilde{X}} \cap X_n = (S+1)\overline{U}^X \cap X_n \subseteq (S+1)V,$$

hence $\mathscr{F}_x$ has a basis in $(S+1)V$, and since $\mathscr{F}_x$ converges to $x$ in $X_m$ this yields $x \in \overline{(S+1)V}^{X_m} \subseteq (S+2)V$ for every $x \in B$. □

One can easily show that under the assumption of theorem 6.10 the spectrum is even boundedly retractive and hence also compactly and sequentially retractive.

We finish this chapter with a "quantitative" sufficient condition for local closed neighbourhoods.

**Proposition 6.11** *If an inductive spectrum $(X_n)_{n\in\mathbb{N}}$ satisfies*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, U \in \mathscr{U}_0(X_m)\ \exists\, V \in \mathscr{U}_0(X_n)\ \forall\, \varepsilon > 0$$

$$\exists\, W \in \mathscr{U}_0(X) \quad W \cap V \subseteq \varepsilon U$$

*(i.e. each continuous seminorm on $X_m$ induces on some neighbourhood of $X_n$ a topology which is coarser than the limit topology) then $(X_n)_{n\in\mathbb{N}}$ has local closed neighbourhoods.*

**Proof.** Given $U \in \mathscr{U}_0(X_n)$ we choose $V \in \mathscr{U}_0(X_m)$ as in the proposition with $V \subseteq \frac{1}{2}U$. For $x \in \overline{V}^X \cap X_n$ there are $S > 0$ with $x \in SV$ and $W \in \mathscr{U}_0(X)$ with $W \cap V \subseteq \varepsilon U$ where $\varepsilon = \frac{1}{2(S+1)}$. Then

$$x \in SV \cap \overline{V}^X \subseteq SV \cap (V + W) \subseteq V + (W \cap (S+1)V)$$

$$\subseteq V + (S+1)(W \cap V) \subseteq V + \frac{1}{2}U \subseteq U.$$

□

Any inductive spectrum which satisfies the assumption of theorem 6.1 also satisfies the condition of the proposition above, hence it has has local closed neighbourhoods. Since for inductive spectra of metrizable spaces the condition in 6.1 is even a characterization of acyclicity, theorem 6.10 covers corollary 6.5.

There are however acyclic inductive spectra (even of $(LB)$-spaces) which do not satisfy the condition of 6.1. Such spectra are constructed by Dierolf, Frerick and S. Müller in [48]. Nevertheless, those inductive spectra are covered by proposition 6.11, and we do not know any acyclic inductive spectrum without local closed neighbourhoods.

# 7

# The duality functor

This final chapter is concerned with the problem when the transposed map of a homomorphism in the category of locally convex spaces is again a homomorphism. Let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an exact sequence in the category of locally convex spaces. The Hahn-Banach theorem implies that the dual sequence

$$0 \longrightarrow Z' \xrightarrow{g^t} Y' \xrightarrow{f^t} X' \longrightarrow 0$$

is exact as a sequence of vector spaces, but if all duals are endowed with the strong topology neither $f^t$ nor $g^t$ must be a homomorphism. Let $D$ be the contravariant functor assigning to a locally convex space $X$ its strong dual $X'_\beta$ and to $f : X \longrightarrow Y$ the transposed map. Then an exact complex

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is transformed into an acyclic complex

$$0 \longrightarrow D(Z) \longrightarrow D(Y) \longrightarrow D(X) \longrightarrow 0$$

To measure the exactness of this complex Palamodov used the functors $H_M$ introduced in section 2.2. For any non-empty set $M$ we define the covariant functor $D_M = H_M \circ D$ from $\mathcal{LCS}$ to the category of vector spaces. Explicitely, to a locally convex space $X$ we assign $D_M(X) = \mathrm{Hom}(X'_\beta, \ell^\infty_M)$, and for a morphism $f : X \longrightarrow Y$ the linear map

$$f^* = D_M(f) : \mathrm{Hom}(X'_\beta, \ell^\infty_M) \longrightarrow \mathrm{Hom}(Y'_\beta, \ell^\infty_M)$$

is defined by $T \mapsto T \circ f^t$. From theorem 2.2.2 we deduce that for an exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

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the dual sequence

$$0 \longrightarrow X' \xrightarrow{g^t} Y' \xrightarrow{f^t} X' \longrightarrow 0$$

is left exact at $Y'$ (i.e. $g^t$ is open onto its range) or right exact at $Y'$ (i.e. $f^t$ is open) respectively if and only if for every set $M \neq \emptyset$ the complex

$$0 \longrightarrow D_M(X) \xrightarrow{f^*} D_M(Y) \xrightarrow{g^*} D_M(Z) \longrightarrow 0$$

is exact at $D_M(Z)$ or exact at $D_M(Y)$, respectively. (We note that the complex is always exact at $D_M(X)$, i.e. $D_M$ is a semi-injective functor.) The exactness at $D_M(Z)$ and $D_M(Y)$ are measured by $D^1_M(X)$ and $D^+_M(X)$, respectively. Let us recall the definitions of $D^1_M$ and $D^+_M$. If

$$0 \longrightarrow X \xrightarrow{i} I_0 \xrightarrow{i_1} I_1 \xrightarrow{i_2} \dots$$

is any injective resolution of $X$ we have

$$D^+_M(X) = \ker D_M(i_0)/\operatorname{im} D_M(i) \text{ and}$$

$$D^1_M(X) = \ker D_M(i_1)/\operatorname{im} D_M(i_0).$$

From this and theorem 2.2.2 we deduce $D^+_M(X) = 0$ for every set $M$ iff $i^t : I_0' \longrightarrow X'$ is open, and $D^1_M(X) = 0$ for every set $M$ if $i_0^t : I_1' \longrightarrow I_0'$ is open onto its range. We note that $i_0$ factorizes as $i_0 = j \circ q$ where $q : I_0 \longrightarrow I_0/\operatorname{im} X$ is the quotient map and $j : I_0/\operatorname{im} X \longrightarrow I_1$ is a topological embedding. As we have remarked in section 2.1, $D^+_M(I_0/\operatorname{im} X) = 0$ holds for every set $M \neq \emptyset$ and from what we have said above we obtain that $j^t$ is open. Therefore, $i_0^t = q^t \circ j^t$ is open onto its range if and only if $q^t$ is open onto its range (hence a monohomomorphism). This leads to the following characterization.

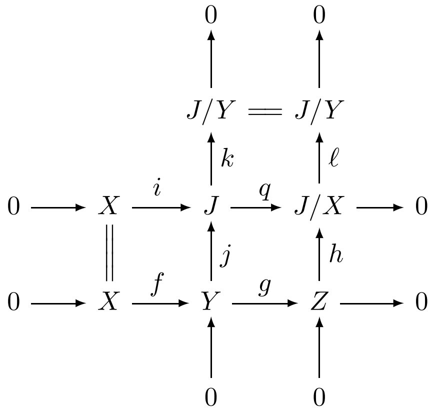
**Theorem 7.1** *Let $X$ be a locally convex space.*

1. *$D^+_M(X) = 0$ for every set $M \neq \emptyset$ if and only if for every short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ the transposed $f^t : Y'_\beta \longrightarrow X'_\beta$ is open.*
2. *$D^1_M(X) = 0$ for every set $M \neq \emptyset$ if and only if for every short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ with $D^+_N(Y) = 0$ for all $N \neq \emptyset$ the transposed $g^t : Z'_\beta \longrightarrow Y'_\beta$ is open onto its range.*

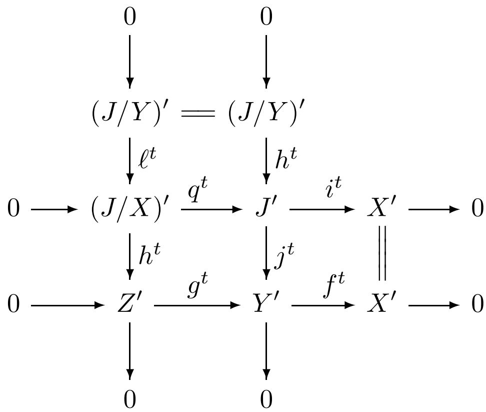
**Proof.** The necessity of $D^+_M(X) = 0$ and $D^1_M(X) = 0$ follows from the remarks above if we choose $Y$ as an injective object. Let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an exact sequence and $j : Y \longrightarrow J$ a monohomomorphism into an injective object. We obtain the following commutative diagram



with exact rows and columns. The dual diagram



is still commutative and algebraically exact. If now $D_M^+(X) = 0$ holds for all sets $M \neq \emptyset$ then $i^t$ is open. Hence, $i^t((j^t)^{-1}(U)) = f^t(U) \in \mathscr{U}_0(X'_\beta)$ for every $U \in \mathscr{U}_0(Y'_\beta)$ which proves the first part of the theorem. If $D_M^1(X) = D_M^+(Y) = 0$ for all $M \neq \emptyset$ the remarks preceding the theorem imply that $q^t$ is open onto its range and $j^t$ is open. For every $U \in \mathscr{U}_0(Z'_\beta)$ there is $V \in \mathscr{U}_0(J'_\beta)$ with $V \cap \operatorname{im} q^t \subseteq q^t((h^t)^{-1}(U))$. Let us show

$$j^t(V) \cap \operatorname{im} g^t \subseteq g^t(U).$$

Given $v \in V$ and $z \in Z'$ with $j^t(v) = g^t(z)$ we choose $a \in (J/X)'$ with $h^t(a) = z$. Since $j^t(q^t(a)) = g^t(h^t(a)) = j^t(v)$ there is $b \in (J/Y)'$ with $k^t(b) = v - q^t(a)$, thus $v = q^t(c)$ which implies

$$j^t(v) = j^t(q^t(c)) = g^t(h^t(c)) \in g^t(U).$$

This completes the proof of the second part. □

From now on, we will write $D^+(X) = 0$ or $D^1(X) = 0$ as an abbreviation for $D_M^+(X) = 0$ or $D_M^1(X)$ for all sets $M \neq \emptyset$.

We do not know whether the extra assumption $D^+(Y) = 0$ in the second part of the theorem above can be dropped (although this is claimed in [50, page 44]). Below we will see that this is the case if $X$ is a Fréchet space. But let us first give a convenient characterization of $D^+(X) = 0$ which is again due to Palamodov [50, theorem 8.1]. We denote by $X'_i$ the inductive dual of $X$, i.e. $X'$ endowed with locally convex topology having

$$\left\{ \Gamma\Big( \bigcup_{U \in \mathscr{U}_0(X)} \varepsilon_U U^\circ \Big) : \varepsilon_U > 0 \right\}$$

as a basis of the 0-neighbourhood filter.

**Theorem 7.2** *A locally convex space $X$ satisfies $D^+(X) = 0$ if and only if $X'_\beta = X'_i$.*

**Proof.** As we have seen in the proof of 2.2.1, $X$ can be embedded into an injective object $I = \prod_{\alpha \in J} X_\alpha$ with semi-normed spaces $X_\alpha$ (in fact, all but one of these spaces are some $\ell^\infty_{M_\alpha}$ and one space is $\overline{\{0\}}^X$). The transposed of the embedding $f = (f_\alpha)_{\alpha \in J} : X \longrightarrow I$ is

$$\bigoplus_{\alpha \in J} X'_\alpha \longrightarrow X', \ (\varphi_\alpha)_{\alpha \in J} \longmapsto \sum_{\alpha \in I} \varphi_\alpha \circ f_\alpha$$

from which we obtain that $X'_i$ is the quotient space

$$\Big( \prod_{\alpha \in J} X_\alpha \Big)' / X^\circ \cong \bigoplus_{\alpha \in J} X'_\alpha / X^\circ.$$

Therefore, $D^+(X) = 0$ iff $X'_\beta = (\prod_{\alpha \in J} X_\alpha)'/X^\circ$ iff $X'_i = X'_\beta$. □

If all bounded sets in the strong dual of a locally convex space $X$ are equicontinuous (i.e. $X$ is quasi-barrelled) then $X'_i$ is the associated bornological topology of $\beta(X', X)$. Hence, for a quasi-barrelled space $X$ we have $D^+(X) = 0$ if and only if $X'_\beta$ is bornological, and by a classical result of Grothendieck [31], for a metrizable space $X$ this happens iff $X'_\beta$ is barrelled, i.e. $X$ is distinguished. Moreover, a theorem of L. Schwartz [56] says that the strong dual of a complete Schwartz space is always bornological (this result is also contained e.g. in [45, theorem 24.23]).

The definition of $D^1(X)$ involves equicontinuous families in the bidual of the locally convex space $X$ which may be difficult to handle. Sometimes it is easier to use the well-known fact that the transposed of a quotient map is a homomorphism iff $q$ lifts bounded sets with closures, i.e. for every bounded set $A \subseteq Z$ there is a bounded set $B \subseteq Y$ with $A \subseteq \overline{q(B)}$ (this follows easily from $(q^t)^{-1}(B^\circ) = q(B)^\circ = \overline{q(B)}^\circ$). As we have already noted in section 3.3

the lifting of bounded sets (without closure) is closely related to the functor $\ell^\infty_M$ assigning to a locally convex space $X$ the vector space

$$\ell^\infty_M(X) = \big\{(x_i)_{i\in M} \in X^M : \{x_i : i \in M\} \text{ is bounded}\big\}$$

and to a morphism $f : X \longrightarrow Y$ the map $f_M : \ell^\infty_M(X) \longrightarrow \ell^\infty_M(Y)$ defined by $(x_i)_{i\in M} \mapsto (f(x_i))_{i\in M}$. These covariant functors are easily seen to be injective and to avoid too many superscripts we denote the derived functors by $L^n_M$, $n \in \mathbb{N}_0$. Theorem 2.1 gives that a locally convex space $X$ satisfies $L^1_M(X) = 0$ for a set $M$ iff for every exact sequence

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{q} Z \longrightarrow 0$$

the sequence

$$0 \longrightarrow \ell^\infty_M(X) \longrightarrow \ell^\infty_M(Y) \xrightarrow{q_M} \ell^\infty_M(Z) \longrightarrow 0$$

is exact which means that $q$ lifts all bounded sets with cardinality less or equal than that of $M$.

It is well-known that quasinormability plays an important role for duality. We will see below that also the “dual” property is helpful in this respect. Let us recall that a locally convex space satisfies the strict Mackey condition (SMC) if its topology coincides on every bounded set with the topology induced by the Minkowski functional of another bounded set.

This condition was introduced by Grothendieck [31], and it is easily seen that it is stable under countable products and subspaces, in particular, every metrizable space satisfies (SMC).

**Lemma 7.3** *Let* $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ *be an exact sequence of locally convex spaces such that* $q$ *lifts bounded sets with closure and* $Z$ *is metrizable. Then*

$$\forall\, U \in \mathscr{U}_0(Y)\ \exists\, V \in \mathscr{U}_0(Z)\ \forall\, A \in \mathscr{B}(Z)\ \exists\, B \in \mathscr{B}(Y)$$
$$A \cap V \subseteq \overline{g(B\cap U)}^{Z}.$$

**Proof.** We assume that the condition fails for some $U \in \mathscr{U}_0(Y)$ and take a decreasing basis $(V_n)_{n\in\mathbb{N}}$ of $\mathscr{U}_0(Z)$. Then there are $A_n \in \mathscr{B}(Z)$ such that $A_n \cap V_n \not\subseteq \overline{g(B\cap U)}$ for all $n \in \mathbb{N}$ and all $B \in \mathscr{B}(Y)$. The set $A = \bigcup_{n\in\mathbb{N}} (V_n \cap A_n)$ is bounded in $Z$ and since $Z$ satisfies (SMC) there is $D \in \mathscr{B}(Z)$ whose Minkowski functional induces the topology of $Z$ on $A$. We take $B \in \mathscr{B}(Y)$ with $D \subseteq \overline{g(B)}$ and $\varepsilon > 0$ with $\varepsilon B \subseteq U$. Then there is $n \in \mathbb{N}$ with $V_n \cap A \subseteq \varepsilon D$, hence

$$A_n \cap V_n \subseteq A \cap V_n \subseteq \varepsilon D \subseteq \overline{g(\varepsilon B)} = \overline{g(\varepsilon B \cap U)} = \overline{g(B\cap U)},$$

a contradiction. □

Together with the Schauder lemma this simple result gives the next theorem of Bonet and Dierolf [8].

**Theorem 7.4** *Let* $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ *be an exact sequence of Fréchet spaces. Then* $g^t$ *is open onto its range if and only if* $g$ *lifts bounded sets.*

**Proof.** We have to show that $g_M : \ell_M^\infty(Y) \longrightarrow \ell_M^\infty(Z)$ is surjective for every set $M \neq \emptyset$ whenever $g$ lifts bounded sets with closure. If we endow $\ell_M^\infty(Z)$ with the locally convex topology having

$$\{\ell_M^\infty(U) := U^M \cap \ell_M^\infty(Z) : U \in \mathscr{U}_0(Z)\}$$

as a basis of the 0-neighbourhoods (and similarly $\ell_M^\infty(Y)$), $g_M$ is a continuous linear map between Fréchet spaces, hence it is surjective if (and only if) it is almost open. Given $U \in \mathscr{U}_0(Y)$ we choose $V \in \mathscr{U}_0(Z)$ according to lemma 7.3, then $\ell_M^\infty(V) \subseteq \overline{g_M(\ell_M^\infty(U))}$. Indeed, given $v = (v_i)_{i\in M} \in \ell_M^\infty(V)$ there is $B \in \mathscr{B}(Y)$ with $V \cap \{v_i : i \in M\} \subseteq \overline{g(B \cap U)}$. Hence, for every $W \in \mathscr{U}(Z)$ there are $u_i \in B \cap U$ with $v_i - g(u_i) \in W$, which gives

$$v = g_M\left((u_i)_{i\in M}\right) + \left(v_i - g(u_i)\right)_{i\in M} \in g_M\left(\ell_M^\infty(U)\right) + \ell_M^\infty(W).$$

This proves that $g_M$ is almost open, hence open by the Schauder lemma, and therefore $g_M$ is surjective. □

Now, we easily obtain the following result (the equivalence of 1, 2 and 4 is due to Palamodov [50, theorem 8.2], and the equivalence of conditions 3 and 4 was proved by Merzon [46]).

**Theorem 7.5** *For a Fréchet space* $X$ *the following four conditions are equivalent.*

1. $D^1(X) = 0$.
2. *For every exact sequence* $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ *of locally convex spaces the sequence* $0 \longrightarrow Z'_\beta \xrightarrow{g^t} Y'_\beta \xrightarrow{f^t} X'_\beta \longrightarrow 0$ *is exact.*
3. *For every exact sequence* $0 \longrightarrow X \longrightarrow Y \xrightarrow{g} Z \longrightarrow 0$ *of locally convex spaces* $g$ *lifts bounded sets.*
4. $X$ *is quasinormable.*

**Proof.** As we have noted in 2.2 we can embed $X$ into an injective object $I$ which is a Fréchet space. If $q : I \longrightarrow I/X$ is the quotient map we have $D^1(X) = 0$ iff $q^t$ is open onto its range iff $q$ lifts bounded sets with closure iff $q$ lifts bounded sets iff $L_M^1(X) = 0$ for all sets $M \neq \emptyset$. From this we obtain that the first and third condition are equivalent and that both are equivalent to the property that for every exact sequence

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{g} Z \longrightarrow 0$$

the transposed $g^t$ is a homomorphism. Let now $\mathscr{X} = (X_n, \varrho_m^n)$ be a reduced spectrum of Banach spaces with $\mathrm{Proj}\mathscr{X} = X$. Then $L_M^1(X) = 0$ for every $M \neq \emptyset$ implies that the canonical map $\Psi_{\mathscr{X}} : \prod_{n\in\mathbb{N}} X_n \longrightarrow \prod_{n\in\mathbb{N}} X_n$ lifts bounded sets which yields that $X$ is quasinormable by theorem 3.3.13. Moreover, if $X$ is quasinormable the spectrum $\mathscr{X}$ satisfies the hypothesis of 3.3.14, hence $\Psi_{\mathscr{X}}$ lifts bounded sets and from the exactness of the complex

$$0 \longrightarrow \ell_M^\infty(X) \longrightarrow \ell_M^\infty(\prod_{n\in\mathbb{N}} X_n) \longrightarrow \ell_M^\infty(\prod_{n\in\mathbb{N}} X_n) \longrightarrow L_M^1(X) \longrightarrow 0$$

we deduce $L_M^1(X) = 0$ for every set $M \neq \emptyset$. Thus, we have proved the equivalence of the first and fourth condition. It remains to show that in the second condition $f^t$ is open whenever $X$ is quasinormable. But it is well-known that quasinormable Fréchet spaces are distinguished (a more general will be proved in 7.7 below), hence $D^+(X) = 0$ which implies that $f^t$ is open. □

Motivated by this result, Palamodov [50, §12.5] asked whether $D^+(X) = 0$ and $D^1(X) = 0$ hold for every quasinormable locally convex space. The answer to the second question is (nowadays) easy: if $X$ is any closed subspace of the space $\mathscr{D}$ of test functions on $\mathbb{R}$ which is not a limit subspace (for instance, such examples are contained in Palamodov's own article [50]), then $X$ is a Schwartz space, hence quasinormable, but the quotient map $q : \mathscr{D} \to \mathscr{D}/X$ does not lift bounded sets with closure (which is the same as the lifting without closure since the bounded sets in $\mathscr{D}$ are compact) because otherwise $\mathscr{D}/X$ would be a regular (LF)-space whose steps are Fréchet-Montel spaces and by corollary 6.7 $\mathscr{D}/X$ would then be acyclic contradicting the choice of $X$. Since $\mathscr{D}'_\beta$ is clearly bornological (this follows either from theorem 3.3.4 or Schwartz's theorem mentioned after 7.2) theorem 7.1 implies $D^1(X) \neq 0$.

The answer to the first question is also negative. In fact, in [9] Bonet, Dierolf, and the author constructed a strict projective spectrum $\mathscr{X}$ of complete (LB)-spaces (whose projective limit is then quasinormable because of theorem 3.3.14) such that $(\mathrm{Proj}\mathscr{X})'_\beta$ is not bornological (it is not even countably quasi-barrelled). Theorem 7.2 implies $D^+(\mathrm{Proj}\mathscr{X}) \neq 0$.

Instead of reproducing this example we would like to give a general sufficient condition for the exactness of dual complexes (this result is also contained in [9]). The following lemma is taken from [45, 26.9 and 10]. One might call the assumption a "topological lifting of bounded sets."

**Lemma 7.6** *Let* $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ *be an exact sequence of locally convex spaces such that*

$$\forall\, A \in \mathscr{B}(Z)\ \exists\, B \in \mathscr{B}(Y)\ \forall\, U \in \mathscr{U}_0(Y)\ \exists\, V \in \mathscr{U}_0(Z)$$

$$A \cap V \subseteq q(B \cap U).$$

*Then* $0 \longrightarrow Z'_\beta \xrightarrow{g^t} Y'_\beta \xrightarrow{f^t} X'_\beta \longrightarrow 0$ *is again exact.*

**Proof.** Taking $U = Z$ it is obvious that the condition of the lemma implies that $g$ lifts bounded sets which yields that $g^t$ is a homomorphism. To show that $f^t$ is open we will first show

$$(*)\ \forall\, C \in \mathscr{B}(Y)\ \exists\, D \in \mathscr{B}(X)\ \forall\, U \in \mathscr{U}_0(Y)\ \exists\, W \in \mathscr{U}_0(Y)$$

$$(C+W) \cap \operatorname{im} f \subseteq f(D) + U.$$

Indeed, given $C \in \mathscr{B}(Y)$ we choose $B \in \mathscr{B}(Y)$ according to $A = g(C)$ and set $D = f^{-1}(C+B) \in \mathscr{B}(X)$. Given $U \in \mathscr{U}_0(Y)$ we choose $V \in \mathscr{U}_0(Z)$ with $A \cap V \subseteq g(B \cap \frac{1}{2}U)$ and $W \in \mathscr{U}_0(Y)$ with $W \subseteq \frac{1}{2}U \cap g^{-1}(V)$. For $y \in (C+W) \cap \operatorname{im} f$ there are $c \in C, w \in W$ with $y = c - w$ which gives $g(c-w) = g(y) = 0$ and thus, $g(c) = g(w) \in A \cap V \subseteq g(B \cap \frac{1}{2}U)$. Hence there is $b \in B \cap \frac{1}{2}U$ with $g(c) = g(w) = g(b)$ which implies

$$y = (c-b) - (w-b) \in (C+B) \cap \ker g + (W + \frac{1}{2}U)$$

$$\subseteq (C+B) \cap \operatorname{im} f + U = f(D) + U.$$

We now show that $f^t$ is open. Given $C \in \mathscr{B}(Y)$ we choose $D \in \mathscr{B}(X)$ as in $(*)$. For each $\varphi \in D^\circ \subset X'$ there is $U \in \mathscr{U}_0(Y)$ with $\varphi \in f^{-1}(U)^\circ$. If $W \in \mathscr{U}_0(Y)$ is chosen according to $(*)$ we obtain

$$\varphi \in D^\circ \cap f^{-1}(U)^\circ \subseteq 2\left(D + f^{-1}(U)\right)^\circ$$

$$= 2\left(f^{-1}(f(D)+U)\right)^\circ \subseteq 2\left(f^{-1}(C+W)\right)^\circ.$$

Thus, if $\Phi : \operatorname{im} f \longrightarrow \mathbb{K}$ is the unique functional with $\Phi \circ f = \varphi$ its absolute value its bounded by 2 on $(C+W) \cap \operatorname{im} f$. Since $C + W \in \mathscr{U}_0(Y)$ the Hahn-Banach theorem gives an extension $\psi \in 2(C+W)^\circ \subseteq 2C^\circ$, thus $\varphi = f^t(\psi) \in 2f^t(C^\circ)$. We have shown $\frac{1}{2}D^\circ \subseteq f^t(C^\circ)$, hence $f^t$ is open with respect to the strong topologies. □

**Theorem 7.7** *Let* $0 \longrightarrow X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z \longrightarrow 0$ *be an exact sequence of locally convex spaces. If $g$ lifts bounded sets and $Z$ satisfies (SMC) then*

$$0 \longrightarrow Z'_\beta \overset{g^t}{\longrightarrow} Y'_\beta \overset{f^t}{\longrightarrow} X'_\beta \longrightarrow 0$$

*is again exact in the category of locally convex spaces.*

**Proof.** We check the condition of the previous lemma. Given $A \in \mathscr{B}(Z)$ we choose $D \in \mathscr{B}(Z)$ whose Minkowski functional induces the topology of $Z$ on $A$, and we choose $B \in \mathscr{B}(Y)$ with $D \subseteq q(B)$. For $U \in \mathscr{U}_0(Y)$ there are $\varepsilon \in (0,1)$ with $\varepsilon B \subseteq U$ and $V \in \mathscr{U}_0(Z)$ with $A \cap V \subseteq \varepsilon D$. This yields

$$A \cap V \subseteq q(\varepsilon B) = q(\varepsilon B \cap U) \subseteq q(B \cap U).$$

□

**Corollary 7.8** *Let* $\mathscr{X} = (X_n, \varrho^n_m)$ *be a projective spectrum of locally complete (LB)-spaces satisfying the strict Mackey condition such that*

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m\ \exists\, B \in \mathscr{B}(X_n)\ \forall\, M \in \mathscr{B}(X_m)$$

$$\exists\, K \in \mathscr{B}(X_k) \quad \varrho^n_m(M) \subseteq \varrho^n_k(K) + B.$$

*Then* $D^+(\mathrm{Proj}\mathscr{X}) = 0$*, i.e.* $(\mathrm{Proj}\,\mathscr{X})'_\beta$ *is an (LF)-space.*

**Proof.** We consider the canonical sequence

$$0 \longrightarrow \mathrm{Proj}\mathscr{X} \overset{i}{\longrightarrow} \prod_{n\in\mathbb{N}} X_n \overset{\Psi}{\longrightarrow} \prod_{n\in\mathbb{N}} X_n \longrightarrow 0.$$

Theorem 3.3.14 implies that $\Psi$ lifts bounded sets, in particular, $\mathrm{Proj}^1\mathscr{X} = 0$, and theorem 3.3.3 yields that the sequence above is exact in the category of locally convex spaces. Theorem 7.7 implies that $i^t : \bigoplus_{n\in\mathbb{N}} X'_n \longrightarrow (\mathrm{Proj}\mathscr{X})'_\beta$ is open hence $(\mathrm{Proj}\mathscr{X})'_\beta$ is ultrabornological. Since $\mathrm{Proj}\mathscr{X}$ is barrelled by 3.3.4 theorem 7.2 implies $D^+(\mathrm{Proj}\mathscr{X}) = 0$, i.e. $(\mathrm{Proj}\,\mathscr{X})'_\beta$ is an (LF)-space. □

We do not know any reasonable condition ensuring $D^1(\mathrm{Proj}\mathscr{X}) = 0$ if $\mathscr{X}$ does not consist of Fréchet spaces. It is shown in [68] that under the continuum hypothesis $D^1(\varphi)$ is not 0. Thus, none of the standard properties of a locally convex space $X$ can easily imply $D^1(X) = 0$.

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